Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE INVARIANTS OF LINEAR DIFFERENTIAL EXPRESSIONS.[1]

By Frank Irwin.

Presented by Maxime Bôcher, April 8, 1908. Received June 9, 1908.

Contents.

I. The adjoint differential expression 5–16
§ 1. Ordinary differential expressions 5–7
§ 2. Partial differential expressions of the second order 7–11
Definition of the *adjoint, M*(v) 8
Condition for a *multiplier* 9
Formulas for coefficients of adjoint 9
Lagrange's Identity 10
If $vL(u) - uN(v) = \sum_i \frac{\partial T}{\partial x_i}$, $N(v) = M(v)$ 10
Conditions for $L(u)$ being self-adjoint 10
Three-term form of Lagrange's Indentity 11
§ 3. Partial differential expressions of the nth order 12–16
Definition of the *adjoint* 13
Condition for a *multiplier* 13
Formulas for coefficients of adjoint 14
Symmetrical formulas for same 14
Conditions for $L(u)$ being $(-1)^n$ times its adjoint 15
Lagrange's Identity 15
II. Change of dependent variable; invariants and covariants; invariants of a differential equation 17–27
§ 4. General properties of invariants and covariants 17–19
Formulas for coefficients of transformed expression 17
Definitions of *invariant, covariant* 18
Every invariant is homogeneous 18
Definition of *weight* 19
Every invariant is the sum of isobaric invariants 19
§ 5. Particular invariants 19–22
Adjoint of transformed is ψ times adjoint 19
The b's are invariants 20
They constitute a complete system 20

---

[1] This paper was accepted in June, 1908, by the Faculty of Arts and Sciences of Harvard University in fulfilment of the requirement of a thesis for the degree of Doctor of Philosophy.

The Wronskian process for deriving invariants 21
Every invariant may be expressed as a function of the following invariants: b, the numerators of $\frac{b_{pq}}{b}$ $\frac{b_{p'q'}}{b}$, . . ., and of their derivatives. 22
§ 6. Particular covariants 22–24
§ 7. Multiplication of $L(u)$ by ϕ; invariants of a differential equation 24–25
Invariants of $L(u)$ for this transformation are invariants of $M(v)$ for $v = \psi \cdot v_1$ 24
Definition of an *invariant of the differential equation* 25
If $I(a) = J(b)$ is one, so is $J(a) = I(b)$ 25
Definition of the invariant *adjoint* to a given invariant of the differential equation 25
§ 8. Invariants of the first and second degree of differential expressions and equations 25–27
The b's are essentially the only linear invariants of $L(u)$ . . . 26
Statement of further results 26
III. Reduction to canonical form 27–39
§ 9. Ordinary differential expressions 27–30
Complete system of invariants of $L(u) = 0$ 29
Every invariant is a function of the invariants I_{n-k}, $I_{n-k,\,l}$ 29
Process for deriving invariants 30
§ 10. Partial differential expressions; conditions for the possibility of the reduction 30–33
The property is invariant 30
Second order 31–32
nth order 32–33
§ 11. Partial differential expressions, continuation; invariants thus suggested 34–39
Results 35
Examples 37
Processes for deriving invariants 39
IV. Change of independent variables; invariants and covariants . . 40–50
§ 12. General properties 40–42
Coefficients of transformed differential expression 40
Definition of *invariant, covariant* 41
Every invariant is isobaric 41
Every invariant is the sum of homogeneous invariants . . . 42
§ 13. Particular invariants and covariants 42–45
$A, \sum_{i,j} A_{ij}\, dx_i\, dx_j, \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$; for second order 43
Generalization of the last 44
Generalization of the invariant $\frac{da}{dx}$ 44
§ 14. Reduction to canonical form of an ordinary differential expression 45–47
Results 46
List of invariants 46
Process for deriving invariants 47

§ 15. The adjoint of the transformed differential expression . . . 47–50
V. Conditions for $\phi \cdot L(u)$ being $(-1)^n$ times its adjoint 50–60
§ 16. The conditions 50–55
The property is invariant 50
Ordinary differential expressions of the second order; Sturm's Normal Form 51
Ordinary differential expressions of the nth order 51
Partial differential expressions of the second order 52–53
Solution of problem for this case 53
Partial differential expressions of the nth order 53–55
§ 17. The covariant $\sum_{i,j} \left(\frac{\partial L_i}{\partial x_j} - \frac{\partial L_j}{\partial x_i}\right) dx_i \, \delta x_j$ 55–60
Particular case: two independent variables 59
List of invariants and covariants 59

THE following paper deals with linear differential expressions, both ordinary and partial, and of all orders. The term "differential expression," as used in these pages, refers, then, always to linear expressions. After an introduction devoted to the theory of the adjoint differential expression, the invariants and covariants of a differential expression under the three transformations which leave its general form unchanged are considered.

The presentation of the introductory matter (I) is, in the main, a reproduction of the substance of lectures by Professor Bôcher in Harvard University, or an extension to expressions of the nth order of matters discussed in those lectures for the second order. The same remark applies to a good part of §§ 4, 5, 7. Acknowledgment of other indebtedness is made in the text. References to Wilczynski are to his Projective Differential Geometry. The name of Lie might be expected to occur more often in a paper on such a subject; it is, however, in obtaining the results recorded in §8 only that I have made use of his methods.

For permission to use the matter referred to above, as well as for most helpful guidance and suggestion in the preparation of this paper throughout, my warmest thanks are due to Professor Bôcher.

I. THE ADJOINT DIFFERENTIAL EXPRESSION.

§ 1. *Ordinary Differential Expressions.*

The first part of this paper deals with the theory of the adjoint differential expression. Let us begin by recalling briefly the facts in the case of an *ordinary* linear differential expression of the nth order. For details, reference may be made to Darboux, Surfaces, book iv, chapter

5, a treatment here followed, or to Wilczynski, who devotes a chapter to the subject. Further, the ordinary differential expression may be looked upon as a special case of the partial differential expression discussed below.

Let, then, our differential expression be

$$L(u) = a_n \frac{d^n u}{dx^n} + a_{n-1} \frac{d^{n-1}u}{dx^{n-1}} + a_{n-2} \frac{d^{n-2}u}{dx^{n-2}} + \ldots + a_0 u. \qquad (1)$$

We *define* as its adjoint the expression

$$M(v) = (-1)^n \frac{d^n(a_n v)}{dx^n} + (-1)^{n-1} \frac{d^{n-1}(a_{n-1}v)}{dx^{n-1}} + (-1)^{n-2} \frac{d^{n-2}(a_{n-2}v)}{dx^{n-2}} + \ldots + a_0 v. \qquad (2)$$

If we write $M(v)$ also as

$$M(v) = b_n \frac{d^n v}{dx^n} + b_{n-1} \frac{d^{n-1}v}{dx^{n-1}} + \ldots + b_0 v,$$

the b's will be given by the following formula:

$$b_{n-k} = (-1)^n \sum_{l=0}^{k} (-1)^l \frac{(n-l)!}{(n-k)!\,(k-l)!} \frac{d^{k-l}a_{n-l}}{dx^{k-l}}. \qquad (3)$$

We may establish next, for any two functions, u, v, *Lagrange's Identity,*

$$vL(u) - uM(v) = \frac{dS}{dx},$$

where S is bilinear in u, v, and their first n–1 derivatives. From this by integration would be obtained a Green's Theorem for the particular differential expression in question. Further, if a relation of the form of Lagrange's Identity,

$$vL(u) - uN(v) = \frac{dT}{dx},$$

exists between two expressions of the nth order, $L(u)$ and $N(v)$, then $N(v)$ is the adjoint of $L(u)$. For we shall have

$$u[N(v) - M(v)] = \frac{d(S-T)}{dx},$$

and therefore $N(v) = M(v)$. This follows from the proposition, the truth of which is obvious:

Lemma. If $N(v)$ be a linear differential expression, and T an expression bilinear in u, v, and their derivatives, and if

$$uN(v) = \frac{dT}{dx},$$

then

$$N(v) = 0.$$

Since Lagrange's Identity may be written

$$uM(v) - vL(u) = \frac{d(-S)}{dx},$$

we infer that $L(u)$ is the adjoint of $M(v)$: the relation between an expression and its adjoint is reciprocal.

A *multiplier* of $L(u)$ is defined to be a function, $v(x)$, such that $vL(u)$ is a derivative of a differential expression of the $(n-1)$st order,

$$vL(u) = \frac{dP}{dx}.$$

The condition that v should be a multiplier of $L(u)$ is that v should satisfy the differential equation $M(v) = 0$. The sufficiency of the condition is obvious from Lagrange's Identity; its necessity follows from an application of the lemma to

$$uM(v) = \frac{d(P-S)}{dx}.$$

For conditions that $L(u)$ should be self-adjoint, when n is even, the negative of its adjoint, when n is odd, that is, $L(u) = (-1)^n M(u)$, see below, page 15. The problem of making $L(u)$ equal to $(-1)^n$ times its adjoint by multiplying it by a suitable function of x will occupy us later.

§ 2. *Partial Differential Expressions of the Second Order.*

We take up next the theory of the adjoint for partial differential expressions, and here a somewhat different order of presentation will be found advantageous. We consider first expressions of the second order.

Let $L(u)$ be such an expression,

$$L(u) = \sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} a_i \frac{\partial u}{\partial x_i} + au. \tag{4}$$

Here we make once for all the convention $a_{ij} = a_{ji}$. Let us inquire as to the condition that a function of the x's should be a multiplier of $L(u)$, the term being defined as follows:

Definition. By a *multiplier* of $L(u)$ is meant a function,

$$v(x_1, \ldots x_m),$$

such that

$$vL(u) = \sum_i \frac{\partial P_i}{\partial x_i}, \tag{5}$$

where the P's are linear differential expressions of the first order.

First suppose that v is such a multiplier. Writing

$$P_i = \sum_j p_{ij} \frac{\partial u}{\partial x_j} + p_i u,$$

we see that we must have

$$2va_{ij} = p_{ij} + p_{ji}, \tag{6a}$$

$$va_i = \sum_j \frac{\partial p_{ji}}{\partial x_j} + p_i, \tag{6b}$$

$$va = \sum_i \frac{\partial p_i}{\partial x_i}. \tag{6c}$$

Operating on the first of these equations with $\frac{\partial^2}{\partial x_i \partial x_j}$, on the second with $-\frac{\partial}{\partial x_i}$, summing and adding to the last equation, the right side cancels out and we have left

$$\sum_{i,j} \frac{\partial^2 (a_{ij} v)}{\partial x_i \partial x_j} - \sum_i \frac{\partial (a_i v)}{\partial x_i} + av = 0.$$

Our assumptions here are that the second derivatives of the a_{ij}'s, the first of the a_i's, that come in question, exist, and, if we desire that property in the coefficients of the equation last written, are continuous. The left side of that equation is, like $L(u)$, a linear differential expression of the second order; we define it to be the adjoint of $L(u)$.

Definition. By the *adjoint* of $L(u)$ we mean the expression

$$M(v) = \sum_{i,j} \frac{\partial^2 (a_{ij} v)}{\partial x_i \partial x_j} - \sum_i \frac{\partial (a_i v)}{\partial x_i} + av. \tag{7}$$

We have proved, then, that a necessary condition that v should be a multiplier of $L(u)$ is that it should satisfy the differential equation $M(v) = 0$.

The condition is also sufficient. For let v be any solution of $M(v) = 0$. Then choose, for instance, the p_{ij}'s for which $i > j$ at pleasure; then the rest of the p_{ij}'s and the p_i's may be determined to satisfy equations (6a) and (6b). Equation (6c) will thereby be satisfied also, and we shall have

$$vL(u) = \sum_i \frac{\partial P_i}{\partial x_i}.$$

For if (6a) and (6b) are satisfied,

$$\sum_{i,j} \frac{\partial^2 (a_{ij} v)}{\partial x_i \partial x_j} - \sum_i \frac{\partial (a_i v)}{\partial x_i} = - \sum_i \frac{\partial p_i}{\partial x_i}.$$

Now since $M(v) = 0$, the left side is equal to $-av$; that is, equation (6c) is satisfied too, as asserted. These considerations show us that the quantities P_i on the right side of (5) are not uniquely determined by v being given. We may state the result just obtained by saying:

Proposition 1. A necessary and sufficient condition that v should be a multiplier of $L(u)$ is that v should satisfy the differential equation $M(v) = 0$.

If we write $M(v)$ in expanded form,

$$M(v) = \sum_{i,j} b_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial v}{\partial x_i} + bv,$$

then the b's, the coefficients of the adjoint, will be given by the formulas

$$\left.\begin{aligned} b_{ij} &= a_{ij}, \\ b_i &= 2 \sum_j \frac{\partial a_{ij}}{\partial x_j} - a_i, \\ b &= \sum_{i,j} \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j} - \sum_i \frac{\partial a_i}{\partial x_i} + a. \end{aligned}\right\} \tag{8}$$

These equations may also be written in symmetrical form,

$$\left.\begin{aligned} \sum_j \frac{\partial b_{ij}}{\partial x_j} - b_i &= - \sum_j \frac{\partial a_{ij}}{\partial x_j} + a_i, \\ \sum_i \frac{\partial b_i}{\partial x_i} - 2b &= \sum_i \frac{\partial a_i}{\partial x_i} - 2a. \end{aligned}\right\} \tag{9}$$

We see thus that if $M(v)$ be the adjoint of $L(u)$, then $L(u)$ is the adjoint of $M(v)$.

Analogous to Lagrange's Identity for ordinary differential expressions we have here too an identity to which we may likewise give that name, holding for any two functions u, v.

Lagrange's Identity. $vL(u) - uM(v) = \sum_i \frac{\partial S_i}{\partial x_i}$,

$$S_i = \sum_j a_{ij}\left(v\frac{\partial u}{\partial x_j} - u\frac{\partial v}{\partial x_j}\right) + \left(a_i - \sum_j \frac{\partial a_{ij}}{\partial x_j}\right) uv.$$

This we readily verify by direct calculation. This identity furnishes, as for ordinary differential expressions, a simple proof of the sufficiency of the condition $M(v) = 0$ for v being a multiplier of $L(u)$. Furthermore we have, here as there, the proposition:

Proposition 2. If between any two differential expressions of the second order, $L(u)$ and $N(v)$, we have an identity of the form of Lagrange's Identity,

$$vL(u) - uN(v) = \sum_i \frac{\partial T_i}{\partial x_i},$$

the T's being bilinear expressions in u, v, and their first derivatives, then $N(v)$ is the adjoint of $L(u)$.

For we get with the help of Lagrange's Identity,

$$u[N(v) - M(v)] = \sum_i \frac{\partial (S_i - T_i)}{\partial x_i};$$

so that u is a multiplier of the differential expression $N(v) - M(v)$, and therefore satisfies the differential equation

$$\text{Adjoint of } [N(v) - M(v)] = 0.$$

But u is any function whatever. Therefore the adjoint of $N(v) - M(v)$, and so $N(v) - M(v)$ itself, is identically zero.

Integration of Lagrange's Identity supplies, as noted for ordinary differential expressions, a Green's Theorem for the expression $L(u)$.

Necessary and sufficient conditions that $L(u)$ should be *self-adjoint* are

$$a_i = \sum_j \frac{\partial a_{ij}}{\partial x_j}, \qquad i = 1, \ldots m. \tag{10}$$

For these are, by (8), the conditions that b_i should equal a_i, and from them follows $b = a$. For the cases, so common in mathematical physics, where the coefficients of the second derivatives in $L(u)$ are *con-*

stants, these conditions reduce to $a_i = 0$. Thus Laplace's equation is self-adjoint.

For self-adjoint differential expressions, the S_i's in Lagrange's Identity reduce to

$$S_i = \sum_{ij} a_{ij} \left(v \frac{\partial u}{\partial x_j} - u \frac{\partial v}{\partial x_j} \right),$$

and that identity may be thrown into the form

$$vL(u) - \sum_i \frac{\partial P_i}{\partial x_i} = uL(v) - \sum_i \frac{\partial Q_i}{\partial x_i},$$

$$P_i = v \sum_j a_{ij} \frac{\partial u}{\partial x_j}, \qquad Q_i = u \sum_j a_{ij} \frac{\partial v}{\partial x_j}. \tag{11}$$

On the other hand we have for $L(u)$, if self-adjoint,

$$L(u) = \sum_i \frac{\partial}{\partial x_i} \left[\sum_j a_{ij} \frac{\partial u}{\partial x_j} \right] + au.$$

On inserting this value of $L(u)$ in Lagrange's Identity above, the left side goes over into

$$v \sum_i \frac{\partial}{\partial x_i} \left[\sum_j a_{ij} \frac{\partial u}{\partial x_j} \right] + auv - \sum_i \frac{\partial}{\partial x_i} \left[v \sum_j a_{ij} \frac{\partial u}{\partial x_j} \right],$$

that is,

$$- \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + auv.$$

Proposition 3. For self-adjoint differential expressions we get a three-term form of Lagrange's Identity,

$$vL(u) - \sum_i \frac{\partial P_i}{\partial x_i} = uL(v) - \sum_i \frac{\partial Q_i}{\partial x_i} = - \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + auv,$$

the P's and Q's being given by (11).

Integration would give a corresponding three-term form of Green's Theorem.

In conclusion, attention may be called to the fact that most of the above can be made to apply directly (1) to ordinary differential expressions of the second order, (2) to differential expressions of the first order, by simply putting the proper coefficients in $L(u)$ equal to zero. A similar remark is in order for the developments of the next paragraph. We note that an expression of the first order can never be self-adjoint, but may be the negative of its adjoint.

§ 3. *Partial Differential Expressions of the nth Order.*[2]

For the general case, partial differential expressions of the nth order, we shall content ourselves with considering differential expressions in two independent variables. The formulas themselves suggest what the extension to the case of a greater number of variables will be, and this suggestion leads throughout to the correct formulas for the latter case. We emphasize once for all this remark, which applies to the whole of the rest of this paper.

We make use here of the following notation:

$$L(u) = \sum_{k=0}^{n}\sum_{p=0}^{n-k} \frac{(n-k)\,!}{p\,!\,q\,!}\, a_{pq} \frac{\partial^{n-k}u}{\partial x^p \partial y^q}, \tag{12}$$

q being defined by $p + q = n - k$; while the subscripts of any a denote respectively the number of differentiations with regard to x, y in the derivative of u to which that coefficient is attached. We may pass from this notation to that employed for the second order by writing, as subscripts, p ones and q twos.

We inquire first, as for expressions of the second order, as to the existence of *multipliers* of $L(u)$, that is of functions, v, such that

$$vL(u) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}, \tag{13}$$

where P, Q are linear differential expressions of the $(n-1)$st order,

$$P = \sum_{k=0}^{n-1} \sum_{p=0}^{n-1-k} P_{pq} \frac{\partial^{n-1-k}u}{\partial x^p \partial y^q}, \qquad p + q = n - 1 - k,$$

with a similar expression for Q. If v is to be such a multiplier we must have

$$\frac{n\,!}{p\,!\,q\,!} v a_{pq} = P_{p-1,\,q} + \quad \text{a term coming from } \frac{\partial Q}{\partial y}, \quad \begin{array}{l} p + q = n, \\ p = 1, 2, \ldots n, \end{array}$$

$$v a_{0n} = \quad \text{a term coming from } \frac{\partial Q}{\partial y};$$

---

[2] See Darboux, Surfaces, book iv, chapter 4, and, for the second order, chapter 2 of an article by du Bois-Reymond in Crelle, vol. 104 (1889). Darboux makes use, to obtain the condition for a multiplier, of a very general formula, of which we here deduce the special case we require.

$$\frac{(n-k)\,!}{p\,!\,q\,!} v a_{pq} = P_{p-1,q} + \frac{\partial P_{pq}}{\partial x} + \text{terms coming from } \frac{\partial Q}{\partial y}, \quad \begin{array}{l} p+q=n-k, \\ p=1,2,\ldots(n-k), \end{array}$$

$$v a_{0,\,n-k} = \frac{\partial P_{0,\,n-k}}{\partial x} + \text{terms coming from } \frac{\partial Q}{\partial y},$$

$$k = 1,\ 2,\ \ldots\ (n-1);$$

$$v a_{00} = \frac{\partial P_{00}}{\partial x} + \text{a term coming from } \frac{\partial Q}{\partial y}.$$

Operate on each of these equations with

$$(-1)^{n-k} \frac{\partial^{n-k}}{\partial x^p \partial y^q}, \qquad k = 0,\ 1,\ \ldots\ n,$$

and add. On the left we get the expression

$$M(v) = \sum_{k=0}^{n} \sum_{p=0}^{n-k} (-1)^{n-k} \frac{(n-k)\,!}{p\,!\,q\,!} \frac{\partial^{n-k}(a_{pq} v)}{\partial x^p \partial y^q}, \qquad p + q = n - k. \tag{14}$$

This we define as the *adjoint* of $L(u)$. On the right we get zero. For consider the terms coming from $\frac{\partial P}{\partial x}$. These give

$$\sum_{k=0}^{n-1} \sum_{p=1}^{n-k} (-1)^{n-k} \frac{\partial^{n-k} P_{p-1,q}}{\partial x^p \partial y^q} + \sum_{k=1}^{n} \sum_{p=0}^{n-k} (-1)^{n-k} \frac{\partial^{n-k+1} P_{pq}}{\partial x^{p+1} \partial y^q}.$$

If, in the second sum, we put $p = p' - 1 \;\; k = k' + 1$, it goes over into the negative of the first, and the two cancel each other. Similarly for the terms coming from $\frac{\partial Q}{\partial y}$. A necessary condition, then, that v should be a multiplier of $L(u)$, is that it should be a solution of the differential equation $M(v) = 0$. That the condition is also sufficient, as well as that P and Q in (13) are not uniquely determined when v is given, follows just as for expressions of the second order. As to the former point, we need merely notice that each of the P's itself occurs in one only of the equations above connecting the a's with the P's and the coefficients of Q, in an equation containing the derivative of a P the sum of whose subscripts is greater, that is of a P which may be supposed to have been already determined from the preceding equations.

Writing the adjoint as

$$M(v) = \sum_{k=0}^{n} \sum_{p=0}^{n-k} \frac{(n-k)\,!}{p\,!\,q\,!}\, b_{pq} \frac{\partial^{n-k} v}{\partial x^p \partial y^q}, \qquad p+q=n-k,$$

we may, from (14), calculate the b's in terms of the a's.

Formulas for the coefficients of $M(v)$ in terms of the coefficients of $L(u)$.

$$\left.\begin{aligned}
&p+q=n: \qquad b_{pq} = (-1)^n a_{pq}.\\
&p+q=n-1:\\
&\quad b_{pq} = (-1)^n \left[n \left(\frac{\partial a_{p+1,\,q}}{\partial x} + \frac{\partial a_{p,\,q+1}}{\partial y} \right) - a_{pq} \right].\\
&p+q=n-2:\\
&\quad b_{pq} = (-1)^n \left[\frac{n(n-1)}{2\,!} \left(\frac{\partial^2 a_{p+2,\,q}}{\partial x^2} + 2 \frac{\partial^2 a_{p+1,q+1}}{\partial x \partial y} + \frac{\partial^2 a_{p,\,q+2}}{\partial y^2} \right) \right.\\
&\qquad\qquad \left. - (n-1) \left(\frac{\partial a_{p+1,\,q}}{\partial x} + \frac{\partial a_{p,\,q+1}}{\partial y} \right) + a_{pq} \right].\\
&p+q=n-k;\\
&\quad b_{pq} = (-1)^n \sum_{l=0}^{k} \sum_{i=0}^{k-l} (-1)^l \frac{(n-l)\,!}{(n-k)\,!\,i\,!\,(k-l-i)\,!} \frac{\partial^{k-l} a_{p+i,\,q+k-l-i}}{\partial x^i \partial y^{k-l-i}}.
\end{aligned}\right\} (15)$$

Assuming for the moment the fact, which will be proved presently, that $L(u)$ is the adjoint of $M(v)$, we may obtain symmetrical formulas connecting the a's and b's. For the formulas expressing the a's in terms of the b's may be written down from those just given by simply interchanging the letters a and b throughout. If now, from these two sets of formulas we replace, in the identity

$$(-1)^n\, a_{pq} + (-1)^k\, b_{pq} = (-1)^k\, b_{pq} + (-1)^n\, a_{pq}, \qquad p+q=n-k,$$

on the left side a_{pq}, on the right b_{pq}, by their values in terms of the b's, a's respectively, we obtain the desired symmetrical formula,

$$\sum_{l=0}^{k-1} \sum_{i=0}^{k-l} (-1)^l \frac{(n-l)\,!}{(n-k)\,!\,i\,!\,(k-l-i)\,!} \frac{\partial^{k-l} b_{p+i,\,q+k-l-i}}{\partial x^i \partial y^{k-l-i}} + (-1)^k\, 2 b_{pq}$$

$= (-1)^{n+k}$ times the same function of the a's and their derivatives,

$$p+q=n-k.^{3} \qquad (16)$$

---

[3] It should be pointed out that these formulas are not precisely analogous to those obtained for the second order. For, if we put here $n=2$, $k=2$, we get, using the notation employed for the second order,

The first equation of (15) shows that a differential expression of odd order cannot be self-adjoint, nor one of even order equal to the negative of its adjoint. Let us call a differential expression that is the negative of its adjoint, $L(u) = -M(u)$, anti-self-adjoint. Then we are led to inquire under what conditions a differential expression will be self-adjoint or anti-self-adjoint, $L(u) = (-1)^n M(u)$. Such conditions may be readily deduced from the symmetrical formulas (16). For let

$$b_{pq} = (-1)^n a_{pq}, \qquad p + q = n - l,$$

for $p = 0, 1, \ldots (n - l)$, and for all values of $l < k$, k being a given even integer. Then, on substituting these values in the left member of (16), all the terms but the last on each side cancel, and we have left

$$b_{pq} = (-1)^n a_{pq}, \qquad p + q = n - k,$$

$p = 0, 1, \ldots (n - k)$. Hence, by mathematical induction, we obtain the conditions (which are, of course, necessary):

Proposition 4. Necessary and sufficient conditions that a differential expression should be self-adjoint or anti-self-adjoint, as the case may be, $L(u) = (-1)^n M(u)$, are that the coefficients of the $(n - k)$th derivatives in $L(u)$, should be $(-1)^n$ times the corresponding coefficients of $M(u)$ for all *odd* values of k.

This proposition has already, in effect, been deduced for expressions of the second order; cf. (10), obtained from the second equation of (8) by putting $b_i = a_i$.

Lagrange's Identity. We may deduce for any differential expression a formula similar to what we have called Lagrange's Identity, or rather a great number of such formulas, by the following process:

Take any term of $vL(u)$, $va \dfrac{\partial^k u}{\partial x^p \partial y^q}$, where we now write a simply for the coefficient. We have, to start with,

$$va \frac{\partial^k u}{\partial x^p \partial y^q} = \frac{\partial}{\partial x}\left(va \frac{\partial^{k-1} u}{\partial x^{p-1} \partial y^q} \right) - \frac{\partial (va)}{\partial x} \frac{\partial^{k-1} u}{\partial x^{p-1} \partial y^q}.$$

$$\frac{\partial^2 b_{11}}{\partial x^2} + 2 \frac{\partial^2 b_{12}}{\partial x \partial y} + \frac{\partial^2 b_{22}}{\partial y^2} - \frac{\partial b_1}{\partial x} - \frac{\partial b_2}{\partial y} + 2b = \text{the same function of the } a\text{'s},$$

an equation which differs from the last equation of (9), written for the case of two independent variables, by the presence of the terms in the second derivatives; terms that cancel each other, indeed, on the two sides of the equation just written. The remaining equations, $n = 2$, $k = 1$, given by (16) agree with those of (9).

Treating the second term on the right in the same way, and so on as long as we can, we get finally

$$
\begin{aligned}
va\,\frac{\partial^k u}{\partial x^p \partial y^q} = {} & \frac{\partial}{\partial x}\left(va\,\frac{\partial^{k-1} u}{\partial x^{p-1}\partial y^q}\right) - \frac{\partial}{\partial x}\left(\frac{\partial(va)}{\partial x}\,\frac{\partial^{k-2} u}{\partial x^{p-2}\partial y^q}\right) + \cdots \\
& \pm \frac{\partial}{\partial x}\left(\frac{\partial^{p-1}(va)}{\partial x^{p-1}}\,\frac{\partial^q u}{\partial y^q}\right) \mp \frac{\partial}{\partial y}\left(\frac{\partial^p(va)}{\partial x^p}\,\frac{\partial^{q-1} u}{\partial y^{q-1}}\right) + \cdots \\
& + (-1)^{k-1}\frac{\partial}{\partial y}\left(\frac{\partial^{k-1}(va)}{\partial x^p \partial y^{q-1}}\,u\right) + (-1)^k\,\frac{\partial^k(va)}{\partial x^p \partial y^q}\,u.
\end{aligned}
$$

The last term on the right is the term of $uM(v)$ corresponding to the term of $vL(u)$ chosen. The other terms on the right are derivatives with regard to x or y of expressions bilinear in u, v and their derivatives of order less than n. Applying the same process to all the terms of $vL(u)$, we reach the result:

Lagrange's Identity. For any two functions u, v of x and y,

$$vL(u) - uM(v) = \frac{\partial S}{\partial x} + \frac{\partial T}{\partial y},$$

where S, T are expressions bilinear in u, v and their derivatives of orders up to the $(n-1)$st.

In the process sketched above, there is evidently much that is arbitrary. Thus we might equally well have written

$$va\,\frac{\partial^k u}{\partial x^p \partial y^q} = \frac{\partial}{\partial y}\left(va\,\frac{\partial^{k-1} u}{\partial x^p \partial y^{q-1}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial(va)}{\partial y}\,\frac{\partial^{k-2} u}{\partial x^{p-1}\partial y^{q-1}}\right) + \cdots$$

a choice being offered at each, or at least at many of the steps of the process, of what the next term to be written down shall be; the last term, in any case, being evidently as above $(-1)^k \dfrac{\partial^k(va)}{\partial x^p \partial y^q}\,u$. So that the S and T in Lagrange's Identity are far from being uniquely determined.[4]

Corresponding to proposition 2, page 10, we have here also that if between any two differential expressions there holds an identity of the form of Lagrange's Identity, then each is the adjoint of the other. This justifies the assumption made on page 14 above, that $L(u)$ was the adjoint of $M(v)$.

---

[4] The process employed first above is that suggested by Darboux, Surfaces, **2**, 73, note. His identity numbered (7) on page 72 is derived by some other of the many possible processes.

II. CHANGE OF DEPENDENT VARIABLE; INVARIANTS AND COVARIANTS. INVARIANTS OF A DIFFERENTIAL EQUATION.

§ 4. *General Properties of Invariants and Covariants.*

We take up next the subject of the transformation of a differential expression by change of dependent variable and of the invariants and covariants of such a transformation.

Taking our differential expression in the form (12), let it go over under change of variable, $u = \psi\,(x, y(\cdot\eta$, into a differential expression $\Lambda(\eta)$, with coefficients α. $\Lambda(\eta)$ will be of the nth order, and its coefficients, the α's, may be readily calculated.

Formulas for the coefficients of the transformed differential expression.

$$\left.\begin{aligned}
&p+q=n\colon\quad \alpha_{pq}=a_{pq}\psi.\\
&p+q=n-1\colon\quad \alpha_{pq}=n\left(a_{p+1,\,q}\frac{\partial\psi}{\partial x}+a_{p,\,q+1}\frac{\partial\psi}{\partial y}\right)+a_{pq}\psi.\\
&p+q=n-2\colon\\
&\alpha_{pq}=\frac{n(n-1)}{2\,!}\left(a_{p+2,\,q}\frac{\partial^2\psi}{\partial x^2}+2\,a_{p+1,\,q+1}\frac{\partial^2\psi}{\partial x\partial y}+a_{p,q+2}\frac{\partial^2\psi}{\partial y^2}\right)\\
&\qquad\qquad+(n-1)\left(a_{p+1,\,q}\frac{\partial\psi}{\partial x}+a_{p,q+1}\frac{\partial\psi}{\partial y}\right)+a_{pq}\psi.\\
&p+q=n-k\colon\\
&\alpha_{pq}=\sum_{l=0}^{k}\sum_{i=0}^{k-l}\frac{(n-l)\,!}{(n-k)\,!\,i\,!\,(k-l-i)\,!}\,a_{p+i,\,q+k-l-i}\,\frac{\partial^{k-l}\psi}{\partial x^i\partial y^{k-l-i}}\,.
\end{aligned}\right\}\quad(17)$$

For *ordinary* differential expressions these reduce to

$$\alpha_{n-k}=\sum_{l=0}^{k}\frac{(n-l)\,!}{(n-k)\,!\,(k-l)\,!}\,a_{n-l}\frac{d^{k-l}\psi}{dx^{k-l}},\qquad(18)$$

while for expressions of the second order,

$$L(u)=\sum_{i,j}a_{ij}\frac{\partial^2u}{\partial x_i\partial x_j}+\sum_{i}a_i\frac{\partial u}{\partial x_i}+au,$$

we should get

$$\left.\begin{aligned} \alpha_{ij} &= a_{ij}\psi, \\ \alpha_i &= 2\sum_j a_{ij}\frac{\partial\psi}{\partial x_j} + a_i\psi, \\ \alpha &= \sum_{i,j} a_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j} + \sum_i a_i\frac{\partial\psi}{\partial x_i} + a\psi = L(\psi). \end{aligned}\right\} \qquad (19)$$

It is in the invariants and covariants of this transformation that we shall interest ourselves. These terms we define as follows:

Definition. By an *invariant* of $L(u)$ under the transformation $u = \psi\cdot\eta$ is meant a function, I, of the a's and their derivatives such that the same function of the coefficients of the transformed differential expression is equal, by virtue of the formulas (17), to the original function multiplied by a power of ψ.

$$I \text{ (α's and derivatives)} = \psi^\mu I \text{ (a's and derivatives)},$$

or, in a convenient abbreviated notation,

$$I(\alpha) = \psi^\mu I(a).$$

If $\mu = 0$, we have an absolute, otherwise a relative invariant.

By a *covariant* is meant a function, not only of the a's and their derivatives, but also of u and its derivatives, having an invariant property defined in a manner similar to the above.

We shall concern ourselves wholly with rational, and principally with rational, integral invariants and covariants, and shall always be speaking of the latter, wherever the contrary is not stated or evident from the context. It will be noticed, however, that certain propositions are true for invariants in general.

We begin with some generalities. Every rational invariant is homogeneous. For make the transformation $u = c\cdot\eta$, c being any constant other than zero. The coefficients of the transformed differential expression are each c times the corresponding coefficient of the original expression, $\alpha_{pq} = ca_{pq}$, and the same is true of their derivatives; so that we have: $I(ca) = c^\mu\, I(a)$, which shows that I is homogeneous. We shall, in accordance with the usage in vogue for homogeneous functions in general, speak of μ as the degree of the invariant, even when it is not a polynomial. The corresponding proposition for polynomial covariants is that the degree of any term in the a's and their derivatives minus its degree in u and its derivatives is constant and equal to μ.

We proceed now to attach a *weight* to each of the a's and its derivatives.

Definition. The *weight* or *total weight* of the coefficient a_{pq}, $p + q = n - k$, shall be $n - k$, and the weight of $\frac{\partial^{i+j} a_{pq}}{\partial x^i \partial y^j}$, $n - k - (i + j)$; the *partial weights* with respect to x, y of a_{pq}, $p + q = n - k$, shall be p, q respectively, and of $\frac{\partial^{i+j} a_{pq}}{\partial x^i \partial y^j}$, $p - i$ and $q - j$ respectively. The weight of a product shall be the sum of the weights of its factors; and a polynomial will be said to be isobaric, totally or partially, if all its terms are of the same weight, total or partial. With this definition of weight we have the following proposition:

Proposition 5. An invariant may or may not be isobaric; but if not, it is a mere sum of invariants which are isobaric. This statement may be interpreted with respect either to the total or any one of the partial weights.

We give the proof for the former case. Consider the identity, $I(a) = \psi^{\mu} I(a)$. Let the terms of any given weight, w, in $I(a)$ be represented by $G_w(a)$; and let us fix our attention on the corresponding terms, $G_w(a)$, in $I(a)$. Suppose we attribute, for our immediate purposes, to ψ the weight zero, to its first derivatives the weight minus one, and so on. Then a comparison of the formulas (17) shows that a_{pq}, $p + q = n - k$, is of weight $n - k$ in this system of weights, while any of its derivatives is of weight equal to $n - k$ minus the number of differentiations;. that is, the weights of the a's and their derivatives are the same as those of the corresponding a's and their derivatives. Thus $G_w(a)$ is of weight w, while all the other terms of $I(a)$ are of some other weight; and consequently there can be no cancelling, whole or in part, between those two sets of terms. Therefore, in $I(a) = \psi^{\mu} I(a)$, $G_w(a)$ must be equal to the terms of weight w on the right side of the equation: i. e., $G_w(a) = \psi^{\mu} G_w(a)$, as was to be proved.

This proposition is of service when we are inquiring as to what invariants of a particular degree exist; in which case we may limit the inquiry to isobaric invariants, since all others can be built up from them by addition.

§ 5. PARTICULAR INVARIANTS.

A simple set of invariants is furnished by the coefficients of the adjoint differential expression. That these are invariants follows at once from the proposition:

Proposition 6. The adjoint of the transformed differential expression, $\Lambda(\eta)$, is ψ times the adjoint of the original expression, $L(u)$.

For make, in Lagrange's Identity,

$$vL(u) - uM(v) = \frac{\partial S}{\partial x} + \frac{\partial T}{\partial y},$$

the change of variable, $u = \psi \cdot \eta$. S, T go over into expressions $\overline{S}$, $\overline{T}$ bilinear in η, v and their derivatives of orders up to the $(n-1)$st. This gives us,

$$v \cdot \Lambda(\eta) - \eta \cdot \psi M(v) = \frac{\partial \overline{S}}{\partial x} + \frac{\partial \overline{T}}{\partial y}.$$

But the existence of an identity of this form between the two expressions $\Lambda(\eta)$ and $\psi M(v)$ shows that they are mutually adjoint.

The coefficients of the adjoint, the b's, are then invariants. They are linear in the a's and their derivatives; cf. the formulas for them (15). Moreover, it may be shown that they are essentially the only linear invariants (see below, page 26). In terms of these invariants and their derivatives — which latter, however, are not invariants — every invariant may be expressed rationally and integrally, simply because the a's and their derivatives can be so expressed.

Further, the b's form a complete system of invariants. This phrase we use in the following sense. Two configurations are said to be equivalent with regard to a given set of transformations if it is possible to find a transformation of the set that takes the first over into the second, and another that takes the second over into the first. A complete set of *absolute* invariants is a set such that if two configurations have the invariants in question equal, each to each, then the two are equivalent. In the case before us we have to do with relative invariants.

Proposition 7. The linear invariants, the b's, constitute a *complete system* of invariants; that is to say, if the linear invariants of two differential expressions are proportional, the expressions are *equivalent.*

Let $L(u)$, $\Lambda(\eta)$ be the two differential expressions, $M(v)$, $M_1(v)$ their adjoints. By hypothesis the coefficients of these latter are proportional; that is, each coefficient of $M_1(v)$ is, say, $\theta(x, y)$ times the corresponding coefficient of $M(v)$. Therefore $M_1(v) = \theta \cdot M(v)$. Now make in $L(u)$ the change of variable $u = \theta \cdot \eta$, and let it go over thereby into $\Lambda_1(\eta)$. The adjoint of $\Lambda_1(\eta)$ is, by proposition 6, θ times the adjoint of $L(u)$, that is $\theta \cdot M(v)$, that is, $M_1(v)$. But $M_1(v)$ was the adjoint of $\Lambda(\eta)$; so that $\Lambda_1(\eta)$ and $\Lambda(\eta)$, being each the adjoint of $M_1(v)$, must be identical; $L(u)$ then goes over, under $u = \theta \cdot \eta$ into $\Lambda(\eta)$. *Q. E. D.*

It is of interest to inquire after processes for deriving, from given invariants, other invariants. One such process is differentiation: the derivative, with regard to any one of the independent variables, of an *absolute* invariant is, in its turn, an absolute invariant; for from $I(a) = I(a)$, $\frac{\partial I(\alpha)}{\partial x_i} = \frac{\partial I(a)}{\partial x_i}$ follows. Since the quotient of any two relative invariants of the same degree is an absolute invariant, this process supplies us with a means of deriving, from two such invariants, a third; a result which, since the denominator, and therefore also the numerator, of the derived invariant are themselves invariants, we may state in another form as follows: If I_1, I_2 be any two relative invariants *of the same degree,* μ, then the Wronskian

$$\begin{vmatrix} I_1 & \frac{\partial I_1}{\partial x} \\ I_2 & \frac{\partial I_2}{\partial x} \end{vmatrix}$$

is also an invariant, and is of degree 2μ. We note that this Wronskian process admits of extensions. If, for instance, I_1, I_2, I_3 be three invariants of the same degree, μ, then both

$$\begin{vmatrix} I_1 & \frac{\partial I_1}{\partial x} & \frac{\partial^2 I_1}{\partial x^2} \\ I_2 & \frac{\partial I_2}{\partial x} & \frac{\partial^2 I_2}{\partial x^2} \\ I_3 & \frac{\partial I_3}{\partial x} & \frac{\partial^2 I_3}{\partial x^2} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} I_1 & \frac{\partial I_1}{\partial x} & \frac{\partial I_1}{\partial y} \\ I_2 & \frac{\partial I_2}{\partial x} & \frac{\partial I_2}{\partial y} \\ I_3 & \frac{\partial I_3}{\partial x} & \frac{\partial I_3}{\partial y} \end{vmatrix}$$

are invariants. And in general the following precept may be laid down for deriving invariants. Write down, as the first column of an m-rowed determinant, m invariants of the same degree, μ. Take for the elements of any other column the derivatives, with regard to any given one of the independent variables, of the elements of some preceding column. This independent variable may be different for different columns. The determinant so constructed will be an invariant of degree $m\mu$. The proof consists in writing down the transformed Wronskian, when everything except $\psi^{m\mu}$ times the original Wronskian will be seen to vanish.

In particular we may derive invariants by this Wronskian process from our linear invariants, the b's. For instance, let b stand for any given one of the b's. Then $\frac{\partial b_{pq}}{\partial x} b - b_{pq}\frac{\partial b}{\partial x}$ is an invariant of the second degree. But so also is b^2. Therefore

$$\frac{\partial}{\partial y}\left(\frac{\partial b_{pq}}{\partial x} b - b_{pq}\frac{\partial b}{\partial x}\right)b - 2\left(\frac{\partial b_{pq}}{\partial x} b - b_{pq}\frac{\partial b}{\partial x}\right)\frac{\partial b}{\partial y}$$

is an invariant of the third degree; and so on. These invariants are evidently merely the numerators of the various derivatives of $\frac{b_{pq}}{b}$. With regard to them we have the following proposition:

Proposition 8. Let b be any chosen one of the b's. Then every invariant can be expressed rationally, and save for the possible presence of a power of b as a denominator, integrally, in terms of b and the numerators of $\frac{b_{pq}}{b}, \frac{b_{p'q'}}{b}, \ldots$ and the numerators of the derivatives of $\frac{b_{pq}}{b}, \frac{b_{p'q'}}{b}, \ldots$ all these numerators being themselves rational, integral invariants. The notation chosen for the enunciation refers to the case of partial differential expressions in two independent variables: the proposition is valid in every case.

Let the invariant $I(a)$ be expressed in terms of the b's: $I(a) = J(b)$. Put $u = \frac{1}{b}\cdot\eta$, and let $L(u)$ go over into $\Lambda(\eta)$. Since the adjoint of $\Lambda(\eta)$ is $\frac{1}{b}$ times the adjoint of $L(u)$ we shall get $I(\alpha)$ by substituting in $J(b)$, for $b_{pq}, b_{p'q'}, \ldots$ and their derivatives, $\frac{b_{pq}}{b}, \frac{b_{p'q'}}{b}, \ldots$ and their derivatives. But $I(\alpha) = \frac{1}{b^{\mu}} I(a)$. This gives us

$$J\left(b, b_{pq}, b_{p'q'}, \ldots; \frac{\partial b}{\partial x}, \frac{\partial b_{pq}}{\partial x}, \ldots; \frac{\partial b}{\partial y}, \frac{\partial b_{pq}}{\partial y}, \ldots\right)$$
$$= b^{\mu} J\left(1, \frac{b_{pq}}{b}, \frac{b_{p'q'}}{b}, \ldots; 0, \frac{\partial}{\partial x}\left(\frac{b_{pq}}{b}\right), \ldots; 0, \frac{\partial}{\partial y}\left(\frac{b_{pq}}{b}\right), \ldots\right).$$

As to a determination of all invariants of the second degree, see below, page 26.

§ 6. *Particular Covariants.*

The simple set of covariants which we now go on to deduce will be, apart from such interest as they may possess in themselves, of use to us later in another connection. For *ordinary* differential expressions the $n + 1$ expressions

$$\sum_{l=0}^{k} \frac{(n-l)!}{(n-k)!\,(k-l)!}\, a_{n-l}\, \frac{d^{k-l}u}{dx^{k-l}}, \qquad k = 0, 1, \ldots n,$$

are absolute covariants. Note that for $k = n$ the expression reduces to $L(u)$. This result is simply a translation into terms of differential expressions of the corresponding facts in the case of ordinary differential equations given by Wilczynski, Chapter II, § 2.[5] And what follows is a mere extension to the case of *partial* differential expressions.

The formulas, (17), expressing the α's in terms of the a's may be given a form more advantageous for some purposes by introducing, in the coefficients of $L(u)$, $\Lambda(\eta)$, further binomial coefficients. Let us put $a_{pq} = \frac{n!}{(n-k)!\,k!}\, c_{pq}$, $\quad p + q = n - k$, and, correspondingly, $\alpha_{pq} = \frac{n!}{(n-k)!\,k!}\, \gamma_{pq}$. $\quad L(u)$ thus becomes,

$$L(u) = \sum_{k=0}^{n} \sum_{p=0}^{n-k} \frac{n!}{k!\,p!\,q!}\, c_{pq}\, \frac{\partial^{n-k}u}{\partial x^p \partial y^q}, \qquad p + q = n - k,$$

while the formulas of transformation are

$$\gamma_{pq} = \sum_{l=0}^{k} \sum_{i=0}^{k-l} \frac{k!}{l!\,i!\,(k-l-i)!}\, c_{p+i,\, q+k-l-i}\, \frac{\partial^{k-l}\psi}{\partial x^i \partial y^{k-l-i}}, \qquad p + q = n - k, \qquad (20)$$

formulas in which everything except the subscripts of the c's is *independent of n*. Now let $L_j(u)$ be an expression of the jth order, $j \leqq n$,

$$L_j(u) = \sum_{k=0}^{j} \sum_{p=0}^{j-k} \frac{j!}{k!\,p!\,q!}\, d_{pq}\, \frac{\partial^{j-k}u}{\partial x^p \partial y^q}, \qquad p + q = j - k.$$

If we make the change of variable $u = \psi \cdot \eta$, the coefficients of the transformed expression will, by (20), be given by

$$\delta_{pq} = \sum_{l=0}^{k} \sum_{i=0}^{k-l} \frac{k!}{l!\,i!\,(k-l-i)!}\, d_{p+i,\, q+k-l-i}\, \frac{\partial^{k-l}\psi}{\partial x^i \partial y^{k-l-i}}, \qquad p + q = j - k.$$

Now take any two numbers $\bar{p}$, $\bar{q}$ such that $\bar{p} + \bar{q} = n - j$. If we put $d_{pq} = c_{p+\bar{p},\, q+\bar{q}}$ for all values of p, q such that $p + q \leqq j$, the expression just written for δ_{pq} goes over into

---

[5] These covariants were first given by Cockle, Phil. Mag., **30** (1865); see Bouton's paper in the Amer. Jour. of Math., **21** (1899).

$$\sum_{l=0}^{k}\sum_{i=0}^{k-l}\frac{k\,!}{l\,!\,i\,!\,(k-l-i)\,!}\,c_{p+\bar{p}+i,\,q+\bar{q}+k-l-i}\,\frac{\partial^{k-l}\psi}{\partial x^{i}\partial y^{k-l-i}},$$

that is, by (20), since $p+\bar{p}+q+\bar{q}=n-k$, into $\gamma_{p+\bar{p},\,q+\bar{q}}$. We have then, $\delta_{pq}=\gamma_{p+\bar{p},\,q+\bar{q}}$. Comparing this with the formulas connecting the d's and c's, $d_{pq}=c_{p+\bar{p},\,q+\bar{q}}$, we see that for these values of the d's $L_j(u)$ is an expression that goes over under $u=\psi\cdot\eta$ into the same function of the γ's that $L_j(u)$ itself is of the c's; in other words, it is an absolute covariant. Inserting then these values of the d's in $L_j(u)$, replacing the c's by the a's, and multiplying through by $\frac{n\,!}{j\,!}$, we get the proposition:

Proposition 9. The expressions

$$\sum_{k=0}^{j}\sum_{p=0}^{j-k}\frac{(n-k)\,!}{p\,!\,q\,!}a_{p+\bar{p},\,q+\bar{q}}\,\frac{\partial^{j-k}u}{\partial x^{p}\partial y^{q}}, \qquad p+q=j-k,$$
$$j=0,\,1,\,\ldots\,n,$$

are absolute covariants for $u=\psi\cdot\eta$. Here $\bar{p}$, $\bar{q}$ are *any* given positive integers (or zeros) subject to the condition: $\bar{p}+\bar{q}=n-j$.

For $j=n$, we get $L(u)$ itself.

For $j=0$: $\qquad a_{pq}u, \qquad p+q=n.$

For $j=1$: $\quad n\left(a_{p+1,q}\dfrac{\partial u}{\partial x}+a_{p,q+1}\dfrac{\partial u}{\partial y}\right)+a_{pq}u, \qquad p+q=n-1.$

We note that these covariants are what might, in accordance with a nomenclature we are about to introduce, be called covariants of the differential *equation.*

§ 7. *Multiplication of L(u) by φ; Invariants of a Differential Equation.*

Let us now consider briefly a second transformation to which a differential expression may be subjected, namely, that of multiplying it through by a function ϕ of the independent variable or variables. Represent the coefficients of $\phi\cdot L(u)$ by $\bar{a}$'s. Then we define as an invariant of this transformation an expression $I(a)$, such that $I(\bar{a})=\phi^{\mu}I(a)$.

Between the invariants of $L(u)$ and those of $M(v)$ a simple relation exists.

Proposition 10. An invariant of a differential expression for a multiplication by ϕ is an invariant of its adjoint for change of dependent variable; an invariant for change of dependent variable is an invariant of the adjoint for multiplication by ϕ.

We prove the first part of the proposition. Let $I(a)$ be an invariant of $L(u)$ for multiplication by ϕ; and let $I(a)$, expressed in terms of the b's, be $J(b)$; $I(a) = J(b)$. Let $M(v)$ go over under $v = \phi \cdot v_1$ into $M_1(v_1)$ with coefficients β. Then by proposition 6, page 19, $\phi L(u)$ and $M_1(v_1)$ are mutually adjoint. Therefore $I(a) = J(\beta)$. But

$$I(\bar{a}) = \phi^\mu I(a) = \phi^\mu J(b).$$

Therefore $$J(\beta) = \phi^\mu J(b). \qquad Q.E.D.$$

Expressions $I(a)$ that are invariant not only for change of dependent variable but also for multiplication of $L(u)$ by ϕ it will be natural to speak of as *invariants of the differential equation* $L(u) = 0$.

Now let $I(a) = J(b)$ be any such invariant. By the proposition just proved $J(b)$ is also an invariant of the differential equation $M(v) = 0$. Therefore $J(a)$ is an invariant of $L(u) = 0$; and, the relation between $L(u)$ and $M(v)$ being reciprocal, $J(a) = I(b)$.

Proposition 11. If $I(a) = J(b)$ be an invariant of a differential equation, then so also is $J(a) = I(b)$. We shall call either of two such invariants the *adjoint* of the other.

It is evident that proposition 5, page 19, may be extended to invariants of a differential equation: if not itself isobaric, such an invariant is nothing more than the sum of invariants which are.

As to a complete system of invariants of a differential equation see below, page 29.

§ 8. *Invariants of the First and Second Degree of Differential Expressions and Equations.*

A problem of interest with regard to the invariants of a differential expression or those of a differential equation is that of determining all the invariants of a given degree. The results which I have been able to obtain concern invariants of the first and second degree.

The methods I have employed are as follows. In the first place, as we have seen, we need merely consider invariants isobaric with respect to each independent variable. Next, in the case of invariants of a differential *expression*, we may confine ourselves to such as are homogeneous in *each* b and its derivatives. For if we call the coefficients of $\psi M(v)$ $\bar{b}$'s, we have

$$I(\bar{b}) = \psi^\mu I(b).$$

Now consider the terms of $I(b)$ homogeneous of any given degree in any given one of the b's and its derivatives. The corresponding terms of $I(\bar{b})$ will, since the $\bar{b}$'s are simply the b's multiplied by ψ, be homogeneous of the same degree in the given b and its derivatives; whence it follows that the terms of $I(b)$ in question will themselves constitute an invariant.

The result just obtained enables us to determine at once all linear invariants of a differential expression. For such an invariant may now be taken as containing *one* of the b's and its derivatives only. Then if we consider any of the derivatives of the highest order of that b occurring in $I(b)$, $I(\bar{b})$ will evidently contain uncancelled the same derivative of ψ; so that, if we are to have $I(\bar{b}) = \psi^{\mu} I(b)$, I can contain no derivatives of b at all. (Similar considerations would show that an invariant of any degree, involving one of the b's and its derivatives only, is essentially nothing more than a power of the b.)

Proposition 12. Essentially the only linear invariants of a differential expression are the b's themselves, all others being linear combinations of these invariants.

The general problem, apart from this simple case, may be attacked by the use of Lie's methods, as illustrated in Bouton's paper in the American Journal of Mathematics, vol. 21. The complete system thus obtained of linear partial differential equations, whose solutions are the invariants sought for, takes on, in the case of invariants of a differential expression, a particularly simple form if everything is expressed in terms, not of the a's, but, as above, of the b's and their derivatives. I bring together here the results I have obtained by the use of these and such other methods as suggested themselves, in each particular case, as appropriate.

Proposition 13. Essentially the only invariants of the *second* degree of an *ordinary* differential *expression* are, besides powers and products of the b's themselves, those of the form

$$\frac{db_i}{dx} b_j - b_i \frac{db_j}{dx};$$

of a *partial* differential *expression* in *two* independent variables, those of the form

$$\frac{\partial b_i}{\partial x} b_j - b_i \frac{\partial b_j}{\partial x}, \quad \frac{\partial b_i}{\partial y} b_j - b_i \frac{\partial b_j}{\partial y},$$

b_i, b_j being any two of the b's.

Essentially the only invariant of the *first* degree of an *ordinary* differential *equation* is b_n and essentially the only invariants of the *second* degree are b_n^2 and

$$nb_nb_{n-2} + \frac{n(n-1)}{2}\left(\frac{db_n}{dx}b_{n-1} - b_n\frac{db_{n-1}}{dx}\right) - \frac{n-1}{2}b_{n-1}^2$$

$$= na_na_{n-2} + \frac{n(n-1)}{2}\left(\frac{da_n}{dx}a_{n-1} - a_n\frac{da_{n-1}}{dx}\right) - \frac{n-1}{2}a_{n-1}^2,$$

which is na_n times the invariant called I_{n-2}, (23) below.

Essentially the only invariants of the *first* degree of a *partial* differential *equation* of the *second* order in *two* independent variables are b_{11}, b_{12}, b_{22}, and of the *second* degree, besides powers and products of the b_{ij}'s, those of the form

$$\frac{\partial b_{ij}}{\partial x}b_{kl} - b_{ij}\frac{\partial b_{kl}}{\partial x}, \quad \frac{\partial b_{ij}}{\partial y}b_{kl} - b_{ij}\frac{\partial b_{kl}}{\partial y},$$

invariants which involve the b's with two subscripts only.

III. Reduction to Canonical Form.

§ 9. *Ordinary Differential Expressions.*

Another method of treating the problem of invariants, a method that applies to the case of an *ordinary* differential expression, is to reduce that expression by a suitable change of dependent variable, to what we shall call its *canonical* form, namely, a form in which the coefficient of the $(n-1)$st derivative is zero.

The corresponding investigation for the case of ordinary differential *equations* will be found in Wilczynski, Chapter II, § 2.[6] The treatment of the two cases is, to a large extent, identical; so that what follows is given not so much for its own sake as because a number of the results admit of extension to *partial* differential expressions.

Suppose then we have an ordinary differential expression

$$L(u) = a_nu^{(n)} + \dots + a_0u,$$

accents denoting differentiation. Let it be reduced, by the change of variable $u = \theta\cdot\eta$, to canonical form

$$\Lambda(\eta) = A_n\eta^{(n)} + \qquad A_{n-2}\eta^{(n-2)} + \dots + A_0\eta\cdot$$

---

[6] We note, to avoid confusion, that Wilczynski calls our canonical form semi-canonical. The method is due to Cockle, Philosophical Magazine, **39** (1870); see Bouton's paper in the Amer. Jour. of Math., **21** (1899).

We see, from (18), that to have $A_{n-1} = 0$, θ must satisfy the equation

$$na_n\theta' + a_{n-1}\theta = 0,$$

or

$$\theta' = -\frac{1}{n}\,\frac{a_{n-1}}{a_n}\,\theta. \tag{21}$$

The other coefficients are given by the formula

$$A_{n-k} = \sum_{l=0}^{k} \frac{(n-l)\,!}{(n-k)\,!\,(k-l)\,!}\,\theta^{(k-l)}a_{n-l}. \tag{22}$$

Substituting herein the values of the derivatives of θ obtained from (21) by differentiation, we find that A_{n-k} is θ times a rational function of the a's and their derivatives, say $A_{n-k} = I_{n-k}(a)\,\theta$. Here we use the letter I because the expressions in question are, in fact, invariants. For let $L(u)$ go over under $u = \psi \cdot u_1$ into $L_1(u_1)$ with coefficients α. Then $L_1(u_1)$ will go over by $u_1 = \frac{\theta}{\psi}\,\eta$ into $\Lambda(\eta)$ above. Since this is a canonical form for $L_1(u_1)$, as well as for $L(u)$, we shall have

$$A_{n-k} = I_{n-k}(\alpha)\frac{\theta}{\psi}.$$

Comparing this with $A_{n-k} = I_{n-k}(a)\theta$, we get

$$I_{n-k}(\alpha) = \psi I_{n-k}(a).$$

The expressions $I_{n-k} = A_{n-k}/\theta$, $k = 0, 2, 3, \ldots n$, are then rational invariants, of the first degree, of the differential expression. Moreover, they are invariants of the differential equation.

For it will be seen from (21) that θ'/θ is the same for $\phi L(u)$ as for $L(u)$ itself, and the same, it is clear, will be true of $\theta^{(k-l)}/\theta$. We see, then, from (22), that I_{n-k}, that is A_{n-k}/θ, formed for $\phi L(u)$ is ϕ times I_{n-k} formed for $L(u)$; or I_{n-k} is an invariant for multiplication by ϕ.

Now, further, suppose that two differential equations, $L(u) = 0$ and $L_1(u_1) = 0$, have these invariants proportional; that is to say, if $L(u)$, $L_1(u_1)$ go over by $u = \theta \cdot \eta$, $u_1 = \theta_1 \cdot \eta$ into canonical forms with coefficients A and $\bar{A}$ respectively, then $A_{n-k}/\theta = \rho(x)\bar{A}_{n-k}/\theta_1$. If now we multiply the former of these canonical forms by $\frac{\theta_1}{\rho\theta}$, it goes over into the latter. We have thus the proposition:

Proposition 14. The expressions

$$I_{n-k} = \frac{A_{n-k}}{\theta}, \quad k = 0, 2, 3, \dots n,$$

where the A's are the coefficients of the canonical form into which $L(u)$ goes over by $u = \theta \cdot \eta$, form a complete system of invariants of an ordinary differential equation; a complete system, that is, in the sense of equivalence, as explained on page 20.

Next let I be any rational invariant, of degree μ, of the differential expression.

$$\theta^{\mu} I(a_n, a_n', \dots a_{n-1}, a_{n-1}', \dots a_{n-k}, \overset{(l)}{a_{n-k}}, \dots)$$
$$= I\,(A_n, A_n', \dots 0, 0, \dots A_{n-k}, \dots \overset{(l)}{A_{n-k}}, \dots)$$

which, since I is homogeneous, is equal to

$$\theta^{\mu} I \left(\frac{A_n}{\theta}, \frac{A_n'}{\theta}, \dots 0, 0, \dots \frac{A_{n-k}}{\theta}, \dots \frac{\overset{(l)}{A_{n-k}}}{\theta}, \dots \right)$$
$$= \theta^{\mu} I\,(I_n, I_{n1}, \dots 0, 0, \dots I_{n-k}, I_{n-k,l}, \dots),$$

if we put $I_{n-k,\,l} = \overset{(l)}{A_{n-k}}/\theta$. The expressions $I_{n-k,\,l}$ are, like I_{n-k}, rational invariants of the first degree. This we shall prove in a moment, and thus get the proposition:

Proposition 15. Every rational invariant of a differential expression under change of dependent variable is a rational function of the rational invariants of the first degree

$$I_{n-k} = \frac{A_{n-k}}{\theta}, \qquad I_{n-k,\,l} = \frac{\overset{(l)}{A_{n-k}}}{\theta},$$

where the A's are the coefficients of the canonical form into which $L(u)$ goes over under $u = \theta \cdot \eta$, and θ satisfies (21).

$$I(a_n, a_n', \dots a_{n-1}, a_{n-1}', \dots a_{n-k}, \dots \overset{(l)}{a_{n-k}}, \dots)$$
$$= I(I_n, I_{n1}, \dots 0, 0, \dots I_{n-k}, \dots I_{n-k,\,l}, \dots).$$

In particular, if I be a polynomial, it is a polynomial in these invariants as well.

We note that

$$I_{n-2} = \frac{n a_n a_{n-2} + \frac{n(n-1)}{2}(a_n' a_{n-1} - a_n a_{n-1}') - \frac{n-1}{2} a_{n-1}^2}{n a_n}. \quad (23)$$

This is the invariant of proposition 13, page 26.

It remains to prove that $I_{n-k,\,l}$ is a rational invariant of the first degree. This may be done by mathematical induction. For I_{n-k} is such an invariant. Suppose, then, that $I_{n-k,\,l}$ is.

$$I_{n-k,\,l+1} = \frac{1}{\theta}\overset{(l+1)}{A}_{n-k} = \frac{1}{\theta}\frac{d}{dx}\left(\overset{(l)}{A}_{n-k}\right) = \frac{1}{\theta}\frac{d}{dx}\left(I_{n-k,\,l}\,\theta\right)$$

$$= I_{n-k,\,l}' - \frac{1}{n}\frac{a_{n-1}}{a_n} I_{n-k,\,l}\,.$$

So that $I_{n-k,\,l+1}$ is rational, and will be an invariant of the first degree by the following proposition:

Proposition 16. If I be an invariant of degree k, then so also is

$$I' - \frac{k}{n}\frac{a_{n-1}}{a_n} I.$$

For it is equal to

$$\frac{1}{a_n}\left[\frac{k}{n}(na_n' - a_{n-1})\, I + a_n I' - k a_n' I\right]$$

$$= \frac{1}{a_n}\left[\frac{k}{n}(na_n' - a_{n-1})\, I + a_n^{k+1}\frac{d}{dx}\left(\frac{I}{a_n^{k}}\right)\right].$$

Here a_n and $na_n' - a_{n-1}$, which is simply $(-1)^n b_{n-1}$, are invariants of the first degree, while I/a_n^k, and therefore its derivative, too, is an absolute invariant. It is apparent that the whole expression is an invariant of degree k.

§ 10. *Partial Differential Expressions: Conditions for the Possibility of Reduction to Canonical Form.*

We pass now to *partial* differential expressions. Here it is not in general possible, as will appear, to reduce the expression, by a change of dependent variable, to canonical form, where now by a canonical form we mean an expression in which the coefficients of *all* the $(n-1)$st derivatives are zero. Let us ask ourselves under what conditions this will be possible. The problem is of interest, not only in itself, but because it will suggest to us certain expressions analogous to the invariants A_{n-k}/θ, to which we were led, in the case of ordinary differential expressions, by the reduction to canonical form; and these expressions will turn out to be, like their prototypes, invariants of the differential equation $L(u) = 0$. We shall also find something analogous to the invariants $\overset{(l)}{A}_{n-k}/\theta$ of the differential expression $L(u)$.

Let us notice first that the *property*, the conditions for whose existence we are seeking, is an invariant property. It is evidently so for a change of dependent variable; and it is so also for a multiplication of

$L(u)$ by ϕ. For if $L(u)$ go over under $u = \theta \cdot \eta$ into a canonical form $\Lambda(\eta)$, $\phi L(u)$ will go over under the *same* transformation into $\phi\Lambda(\eta)$, which is likewise canonical. We shall be inclined, then, to expect that the conditions in question will consist in the vanishing of expressions which are "invariants of the differential equation." And such proves to be the case.

We examine the question first for an expression of the *second* order. Let

$$L(u) = \sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i a_i \frac{\partial u}{\partial x_i} + au$$

go over, by $u = \theta \cdot \eta$, into

$$\Lambda(\eta) = \sum_{i,j} \alpha_{ij} \frac{\partial^2 \eta}{\partial x_i \partial x_j} + \sum_i \alpha_i \frac{\partial \eta}{\partial x_i} + \alpha\eta.$$

If this is to be canonical, we must, by (19), page 18, have

$$2 \sum_j a_{ij} \frac{\partial \log \theta}{\partial x_j} + a_i = 0, \qquad i = 1, 2, \ldots m.$$

If

$$A \equiv \begin{vmatrix} a_{11} & \cdot & \cdot & \cdot & a_{1m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdot & \cdot & \cdot & a_{mm} \end{vmatrix} \neq 0,$$

these equations may be solved for $\frac{\partial \log \theta}{\partial x_i}$, $i = 1, 2, \ldots m$. Note here that A is an invariant of the differential equation. The solution in question will be

$$\frac{\partial \log \theta}{\partial x_i} = - \frac{\begin{vmatrix} a_{11} \cdot \cdot \cdot a_{1,\, i-1} & a_1 & a_{1,\, i+1} \cdot \cdot \cdot a_{1m} \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{m1} \cdot \cdot \cdot a_{m,\, i-1} & a_m & a_{m,\, i+1} \cdot \cdot \cdot a_{mm} \end{vmatrix}}{2A} = \kappa_i, \qquad (24)$$

let us say. Necessary and sufficient conditions that these equations possess a solution $\log \theta$ are

$$\frac{\partial \kappa_i}{\partial x_j} - \frac{\partial \kappa_j}{\partial x_i} = 0, \qquad i, j = 1, 2, \ldots m. \qquad (25)$$

The expression on the left of this last equation is an absolute invariant of the differential equation. For, first, κ_i, κ_j themselves are absolute invariants for a multiplication of $L(u)$ by ϕ. Next, if $L(u)$ go over by any transformation $u = \psi \cdot \eta$ into an expression with coefficients $\bar{a}$, and $\bar{\kappa}_i$ be the same function of the $\bar{a}$'s that κ_i is of the a's, we see without difficulty from (19), page 18, that

$$\bar{\kappa}_i = \kappa_i - \frac{\partial \log \psi}{\partial x_i}; \tag{26}$$

so that

$$\frac{\partial \bar{\kappa}_i}{\partial x_j} - \frac{\partial \bar{\kappa}_j}{\partial x_i} = \frac{\partial \kappa_i}{\partial x_j} - \frac{\partial \kappa_j}{\partial x_i}.$$

The invariant of the differential equation adjoint (proposition 11, page 25) to the invariant just found is $\frac{\partial \lambda_i}{\partial x_j} - \frac{\partial \lambda_j}{\partial x_i}$, if λ_i be the same function of the b's that κ_i is of the a's, that is, by (8), page 9, if

$$\lambda_i = - \frac{\begin{vmatrix} a_{11} \cdots a_{1,i-1} & 2\sum_j \frac{\partial a_{1j}}{\partial x_j} - a_1 & a_{1,i+1} \cdots a_{1m} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{m1} \cdots a_{m,i-1} & 2\sum_j \frac{\partial a_{mj}}{\partial x_j} - a_m & a_{m,i+1} \cdots a_{mm} \end{vmatrix}}{2A}. \tag{27}$$

The difference, $\left(\frac{\partial \lambda_i}{\partial x_j} - \frac{\partial \lambda_j}{\partial x_i}\right) - \left(\frac{\partial \kappa_i}{\partial x_j} - \frac{\partial \kappa_j}{\partial x_i}\right)$, of these two invariants of the differential equation is an invariant that we shall come across later.

Consider next a differential expression of the nth order. If $u = \theta \cdot \eta$ carry it over into a canonical form, we must have

$$n\left(a_{p+1,q}\frac{\partial \log \theta}{\partial x} + a_{p,q+1}\frac{\partial \log \theta}{\partial y}\right) + a_{pq} = 0,$$

$$p + q = n - 1, \qquad p = 0, 1, \ldots (n-1).$$

Conditions necessary and, in general, sufficient for these equations being *algebraically* solvable are that all three-rowed determinants of the matrix

$$\left\|\begin{matrix} a_{1,n-1} & a_{0n} & a_{0,n-1} \\ \cdots & \cdots & \cdots \\ a_{p+1,q} & a_{p,q+1} & a_{pq} \\ \cdots & \cdots & \cdots \\ a_{n0} & a_{n-1,1} & a_{n-1,0} \end{matrix}\right\|$$

should vanish. Any one of these three-rowed determinants

$$I = \begin{vmatrix} a_{p_1+1,\,q_1} & a_{p_1,\,q_1+1} & a_{p_1q_1} \\ a_{p_2+1,\,q_2} & a_{p_2,\,q_2+1} & a_{p_2q_2} \\ a_{p_3+1,\,q_3} & a_{p_3,\,q_3+1} & a_{p_3q_3} \end{vmatrix}, \quad p_i + q_i = n - 1, \tag{28}$$

is an invariant of the differential equation. That it is invariant for $u = \psi \cdot \eta$ is seen at once from the formulas of transformation (17), page 17. The adjoint invariant is

$$J = (-1)^n \cdot \begin{vmatrix} a_{p_1+1,q_1} & a_{p_1,q_1+1} & n\left(\dfrac{\partial a_{p_1+1,q_1}}{\partial x} + \dfrac{\partial a_{p_1,q_1+1}}{\partial y}\right) - a_{p_1q_1} \\ a_{p_2+1,q_2} & a_{p_2,q_2+1} & n\left(\dfrac{\partial a_{p_2+1,q_2}}{\partial x} + \dfrac{\partial a_{p_2,q_2+1}}{\partial y}\right) - a_{p_2q_2} \\ a_{p_3+1,q_3} & a_{p_3,q_3+1} & n\left(\dfrac{\partial a_{p_3+1,q_3}}{\partial x} + \dfrac{\partial a_{p_3,q_3+1}}{\partial y}\right) - a_{p_3q_3} \end{vmatrix}. \tag{29}$$

And $I + (-1)^{n-1}J$ is an invariant of the differential equation that we shall come across later.

The remainder of the treatment is like that of the second order.

$$\frac{\partial \log \theta}{\partial x} = -\frac{\begin{vmatrix} a_{p_1q_1} & a_{p_1,q_1+1} \\ a_{p_2q_2} & a_{p_2,q_2+1} \end{vmatrix}}{n\,A} = \kappa_1, \quad \frac{\partial \log \theta}{\partial y} = -\frac{\begin{vmatrix} a_{p_1+1,q_1} & a_{p_1q_1} \\ a_{p_2+1,q_2} & a_{p_2q_2} \end{vmatrix}}{n\,A} = \kappa_2, \tag{30}$$

$$A = \begin{vmatrix} a_{p_1+1,q_1} & a_{p_1,q_1+1} \\ a_{p_2+1,q_2} & a_{p_2,q_2+1} \end{vmatrix},$$

p_i, q_i being any positive integers such that $p_i + q_i = n - 1$. The condition for a solution is:

$$\frac{\partial \kappa_1}{\partial y} - \frac{\partial \kappa_2}{\partial x} = 0, \tag{31}$$

where the expression on the left is an invariant of the differential equation. If $\bar{\kappa}_1$, $\bar{\kappa}_2$ refer to $\Lambda(\eta)$ into which $L(u)$ goes over under $u = \psi \cdot \eta$,

$$\bar{\kappa}_1 = \kappa_1 - \frac{\partial \log \psi}{\partial x}, \quad \bar{\kappa}_2 = \kappa_2 - \frac{\partial \log \psi}{\partial y}. \tag{32}$$

The invariant adjoint to (31) is $\dfrac{\partial \lambda_1}{\partial y} - \dfrac{\partial \lambda_2}{\partial x}$, where

$$\lambda_1 = -\frac{\begin{vmatrix} n\left(\dfrac{\partial a_{p_1+1,q_1}}{\partial x} + \dfrac{\partial a_{p_1,q_1+1}}{\partial y}\right) - a_{p_1q_1} & a_{p_1,q_1+1} \\ n\left(\dfrac{\partial a_{p_2+1,q_2}}{\partial x} + \dfrac{\partial a_{p_2,q_2+1}}{\partial y}\right) - a_{p_2q_2} & a_{p_2,q_2+1} \end{vmatrix}}{n\,A}, \tag{32a}$$

with a similar expression for λ_2.

§ 11. *Partial Differential Expressions: Invariants suggested by the Reduction to Canonical Form.*

In the case of ordinary differential expressions we have seen (proposition 15, page 29) that A_{n-k}/θ, $k = 0, 2, 3, \ldots n$, are invariants of the equation $L(u) = 0$, the A's being the coefficients of the canonical form derived from $L(u)$ by putting $u = \theta \cdot \eta$, where θ is defined by (21), page 28. Are there any corresponding phenomena in the case of partial differential expressions? In the first place it is clear, and might be proved in the same way, that for such partial differential expressions as can be reduced to a canonical form by $u = \theta \cdot \eta$, where θ is defined by (30), the coefficients of that form divided by θ are invariants of the equation, to use the term in such a sense, for that particular class of differential expressions. But for other differential expressions the proof that these same functions of the a's and their derivatives were invariants of the equation would no longer hold. It turns out, nevertheless, that they are in fact invariants of the differential equation, as we now go on to show.

Let us see just what it is that we wish to prove. Consider the formulas for the $\boldsymbol{a}$'s in terms of the a's,

$$
{}_{pq} = \sum_{l=0}^{k} \sum_{i=0}^{k-l} \frac{(n-l)\,!}{(n-k)\,!\,i\,!\,(k-l-i)\,!}\, a_{p+i,\, q+k-l-i}\, \frac{\partial^{k-l}\psi}{\partial x^{i}\,\partial y^{k-l-i}}, \qquad (17)
$$
$$
p + q = n - k.
$$

Now suppose that we substitute in this formula for ψ and its derivatives θ and its derivatives, $\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}$ being given by (30), that is,

$$
\frac{\partial\theta}{\partial x} = \kappa_1\theta, \qquad \frac{\partial\theta}{\partial y} = \kappa_2\theta,
$$

and the higher derivatives of θ being determined from these formulas by differentiation and the substitution, at each step of the process of differentiation, of $\kappa_1\theta$, $\kappa_2\theta$ for $\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}$ respectively. This rule, it will be noticed, does not completely determine the expressions to be substituted; for we may, to take an instance, in accordance with its directions, substitute for $\frac{\partial^2\psi}{\partial x\partial y}$ either $\frac{\partial\kappa_1}{\partial y}\theta + \kappa_1\frac{\partial\theta}{\partial y}$, that is $\left(\frac{\partial\kappa_1}{\partial y} + \kappa_1\kappa_2\right)\theta$, or else $\frac{\partial\kappa_2}{\partial x}\theta + \kappa_2\frac{\partial\theta}{\partial x}$, that is $\left(\frac{\partial\kappa_2}{\partial x} + \kappa_1\kappa_2\right)\theta$. But this does not matter. We suppose the expressions to be substituted for any given derivative of ψ to be calculated *in any way whatever* in accordance only with the rule above. These expressions, as we see, will be θ times polynomials in κ_1, κ_2 and their derivatives. If, finally, we divide the whole by θ, we get a rational function of the a's and their derivatives, and it is this latter expression that we wish to prove an invariant of the differential equation. What we have to prove, then, may be stated in the proposition:

Proposition 17. The expression

$$\frac{1}{\theta}\sum_{l=0}^{k}\sum_{i=0}^{k-l}\frac{(n-l)\,!}{(n-k)\,!\,i\,!\,(k-l-i)\,!}\,a_{p+i,\,q+k-l-i}\,\frac{\partial^{k-l}\theta}{\partial x^i\partial y^{k-l-i}},\qquad p+q=n-k, \tag{33}$$

is an invariant of the differential equation $L(u)=0$, where the derivatives of θ are obtained from

$$\frac{\partial\theta}{\partial x}=\kappa_1\theta,\qquad \frac{\partial\theta}{\partial y}=\kappa_2\theta \tag{34}$$

by the rule above, and κ_1, κ_2 are defined by (30). It is an invariant of degree one.

First, it is an invariant for a multiplication of $L(u)$ by ϕ. For κ_1, κ_2, and therefore their derivatives also, are absolute invariants for this transformation. So too, then, is any derivative of θ divided by θ; while finally each of these latter expressions is multiplied by an a.

Next we have to prove that our expression (33) is an invariant for $u=\psi\cdot\eta$. To this end let us turn back to the absolute covariants of proposition 9, page 24. If we divide any one of these by $(n-j)\,!\,u$, we get a covariant of the first degree, which, by a change of notation, we may write

$$\frac{1}{u}\sum_{l=0}^{k}\sum_{i=0}^{k-l}\frac{(n-l)\,!}{(n-k)\,!\,i\,!\,(k-l-i)\,!}\,a_{p+i,\,q+k-l-i}\,\frac{\partial^{k-l}u}{\partial x^i\partial y^{k-l-i}},\qquad p+q=n-k. \tag{35}$$

Here we note the close analogy in form with (33). In fact, this covariant may be obtained from the formula (17) for a_{pq}, reproduced on page 34 above, by the substitution for the derivatives of ψ of the corresponding derivatives of u divided by u, just as (35) is obtained from the same formula by the substitution of certain polynomials in κ_1, κ_2, and their derivatives.

Now, since we have parallel with each other

$$\frac{\partial\theta}{\partial x} = \kappa_1\theta \qquad \frac{\partial u}{\partial x} = \frac{\dfrac{\partial u}{\partial x}}{u}u$$

$$\frac{\partial\theta}{\partial y} = \kappa_2\theta \qquad \frac{\partial u}{\partial y} = \frac{\dfrac{\partial u}{\partial y}}{u}u,$$

it is evident that, however, from the formulas on the left, we may calculate the value of any derivative of θ, that value, divided by θ, will be the same function of κ_1, κ_2, and their derivatives, as is the corresponding derivative of u divided by u of $\dfrac{\partial u}{\partial x}\Big/u$, $\dfrac{\partial u}{\partial y}\Big/u$, and their derivatives. And thus we reach the result that our expression (33) is the same function of κ_1, κ_2, and their derivatives, that the covariant (35) is of $\dfrac{\partial u}{\partial x}\Big/u$, $\dfrac{\partial u}{\partial y}\Big/u$, and their derivatives.

From this it follows at once that the former, like the latter, is invariant of degree one. For the two sets of arguments in question are cogredient with each other, since we have seen, (32), page 33, that if $\bar{\kappa}_1$, $\bar{\kappa}_2$ stand for the same functions of the $\bar{a}$'s that κ_1, κ_2 are of the a's, then

$$\bar{\kappa}_1 = \kappa_1 - \frac{1}{\psi}\frac{\partial\psi}{\partial x}, \quad \bar{\kappa}_2 = \kappa_2 - \frac{1}{\psi}\frac{\partial\psi}{\partial y},$$

while parallel with this we have

$$\frac{\dfrac{\partial\eta}{\partial x}}{\eta} = \frac{\dfrac{\partial u}{\partial x}}{u} - \frac{1}{\psi}\frac{\partial\psi}{\partial x}, \quad \frac{\dfrac{\partial\eta}{\partial y}}{\eta} = \frac{\dfrac{\partial u}{\partial y}}{u} - \frac{1}{\psi}\frac{\partial\psi}{\partial y};$$

and this parallelism, of course, extends to the derivatives of the quantities in question. Thus the proof of our proposition is complete.

We see from the formulas for κ_1, κ_2, (30), page 33, that our invariants, if reduced to a common denominator, will be polynomials in the a's, and their derivatives divided by a power of A. These polynomials will then themselves be invariants of the differential equation.

The simplest of our invariants are those derived from a_{pq}, where $p + q = n - 2$. Here we have two invariants,

$$I_1 = \frac{n(n-1)}{2}\left[a_{p+2,q}\left(\frac{\partial\kappa_1}{\partial x}+\kappa_1^2\right)+2a_{p+1,q+1}\left(\frac{\partial\kappa_1}{\partial y}+\kappa_1\kappa_2\right)\right.$$
$$\left.+a_{p,q+2}\left(\frac{\partial\kappa_2}{\partial y}+\kappa_2^2\right)\right]+(n-1)\left[a_{p+1,q}\kappa_1+a_{p,q+1}\kappa_2\right]+a_{pq},$$

and I_2, which differs from I_1 in that it replaces $\frac{\partial\kappa_1}{\partial y}+\kappa_1\kappa_2$ in the coefficient of $a_{p+1,q+1}$ by $\frac{\partial\kappa_2}{\partial x}+\kappa_1\kappa_2$. Thus

$$I_1 - I_2 = n(n-1)a_{p+1,q+1}\left(\frac{\partial\kappa_1}{\partial y}-\frac{\partial\kappa_2}{\partial x}\right).$$

Here, since $p+q=n-2$, $a_{p+1,q+1}$ is an invariant of the differential equation, and the other factor we already know to be such, (31).

In (30) p_i, q_i are subject only to the condition $p_i+q_i=n-1$. We may, by a special choice of these numbers, considerably simplify I_1 and I_2, or rather their sum. For putting $p_1=p+1$, $q_1=q$, $p_2=p$, $q_2=q+1$, we get

$$\kappa_1 = -\frac{\begin{vmatrix} a_{p+1,q} & a_{p+1,q+1} \\ a_{p,q+1} & a_{p,q+2}\end{vmatrix}}{nA},\quad \kappa_2 = -\frac{\begin{vmatrix} a_{p+2,q} & a_{p+1,q} \\ a_{p+1,q+1} & a_{p,q+1}\end{vmatrix}}{nA},$$

$$A = \begin{vmatrix} a_{p+2,q} & a_{p+1,q+1} \\ a_{p+1,q+1} & a_{p,q+2}\end{vmatrix}.$$

This would mean that κ_1, κ_2 had been obtained as solutions of the equations

$$n(a_{p+2,q}\kappa_1 + a_{p+1,q+1}\kappa_2) + a_{p+1,q} = 0,$$
$$n(a_{p+1,q+1}\kappa_1 + a_{p,q+2}\kappa_2) + a_{p,q+1} = 0;$$

from which, by differentiation, we get

$$n\left(a_{p+2,q}\frac{\partial\kappa_1}{\partial x}+a_{p+1,q+1}\frac{\partial\kappa_2}{\partial x}\right)$$
$$= -n\left(\frac{\partial a_{p+2,q}}{\partial x}\kappa_1+\frac{\partial a_{p+1,q+1}}{\partial x}\kappa_2\right)-\frac{\partial a_{p+1,q}}{\partial x},$$

$$n\left(a_{p+1,q+1}\frac{\partial\kappa_1}{\partial y}+a_{p,q+2}\frac{\partial\kappa_2}{\partial y}\right)$$
$$= -n\left(\frac{\partial a_{p+1,q+1}}{\partial y}\kappa_1+\frac{\partial a_{p,q+2}}{\partial y}\kappa_2\right)-\frac{\partial a_{p,q+1}}{\partial y}.$$

With the help of these four equations, $I = \frac{1}{2}(I_1 + I_2)$ reduces to

$$I = -\frac{n(n-1)}{2}\left[\kappa_1\left(\frac{\partial a_{p+2,q}}{\partial x} + \frac{\partial a_{p+1,q+1}}{\partial y}\right) + \kappa_2\left(\frac{\partial a_{p+1,q+1}}{\partial x} + \frac{\partial a_{p,q+2}}{\partial y}\right)\right]$$

$$+ \frac{n-1}{2}\left[\kappa_1 a_{p+1,q} + \kappa_2 a_{p,q+1}\right] - \frac{n-1}{2}\left[\frac{\partial a_{p+1,q}}{\partial x} + \frac{\partial a_{p,q+1}}{\partial y}\right] + a_{pq},$$

$$p + q = n - 2.$$

For ordinary differential expressions this reduces, as it should, to the invariant of the differential equation which we have called, (23), page 29, A_{n-2}/θ or I_{n-2}, if we put, as proper,

$$\kappa_1 = -\frac{a_{n-1}}{n a_n}, \qquad \kappa_2 = 0.$$

For the second order, $n = 2$, m variables, the corresponding invariant is:

$$I = \frac{1}{4A}\left[4\,aA - \sum_{i,j} a_i a_j A_{ij} + 2\sum_{i,j,k} a_i A_{ij}\frac{\partial a_{jk}}{\partial x_k} - 2\,A\sum_{i}\frac{\partial a_i}{\partial x_i}\right],$$

where A_{ij} is the cofactor of a_{ij} in

$$A = \begin{vmatrix} a_{11} & \cdot & \cdot & \cdot & a_{1m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdot & \cdot & \cdot & a_{mm} \end{vmatrix}.$$

This becomes for two variables, $m = 2$,

$$I = \frac{1}{4A}\Bigg[4\,aA - (a_1^2 a_{22} - 2\,a_1 a_2 a_{12} + a_2^2 a_{11})$$

$$+ 2\,(a_1 a_{22} - a_2 a_{12})\left(\frac{\partial a_{11}}{\partial x} + \frac{\partial a_{12}}{\partial y}\right)$$

$$+ 2\,(a_2 a_{11} - a_1 a_{12})\left(\frac{\partial a_{12}}{\partial x} + \frac{\partial a_{22}}{\partial y}\right) - 2\,A\left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y}\right)\Bigg],$$

$$A = a_{11}a_{22} - a_{12}^2.$$

Invariants of a partial differential expression analogous to $\overset{(l)}{A}_{n-k}/\theta$. We have found now invariants of a partial differential equation analogous to the invariants A_{n-k}/θ of an ordinary differential equation. It remains to discover the analogue of $\overset{(l)}{A}_{n-k}/\theta$, which, we remember, was an invariant of the differential *expression*. This merely amounts (see proposition 16, page 30) to an inquiry after a process analogous to the process by which from an invariant I of degree one, or, more generally, of degree k, of an ordinary differential expression, we derived a second, $I' - \frac{k}{n} \frac{a_{n-1}}{a_n} I$. The inquiry is answered by the following proposition:

Proposition 18. If I be an invariant of the kth degree of a partial differential expression, then so also are

$$\frac{\partial I}{\partial x} + k\kappa_1 I,$$

$$\frac{\partial I}{\partial y} + k\kappa_2 I,$$

κ_1, κ_2 being defined by (30), page 33. Further, if $p + q = n - 1$, then

$$a_{p+1,q} \frac{\partial I}{\partial x} + a_{p,q+1} \frac{\partial I}{\partial y} - \frac{k}{n} a_{pq} I$$

is an invariant of degree $k + 1$.

We notice that the first two of these invariants may, with the notation of (30), be written as $\frac{1}{\theta^k} \frac{\partial}{\partial x} (\theta^k I)$, $\frac{1}{\theta^k} \frac{\partial}{\partial y} (\theta^k I)$, just as for ordinary differential expressions the derived invariant may, with the notation of (21), page 28, which corresponds to (30), be written $\frac{1}{\theta^k} \frac{d}{dx} (\theta^k I)$.

Proof. The first of the invariants above, formed for the transformed differential expression, is

$$\frac{\partial}{\partial x} (\psi^k I) + k\bar{\kappa}_1 \psi^k I = k\psi^{k-1} \frac{\partial \psi}{\partial x} I + \psi^k \frac{\partial I}{\partial x} + k \left(\kappa_1 - \frac{1}{\psi} \frac{\partial \psi}{\partial x}\right) \psi^k I$$

$$= \psi^k \left(\frac{\partial I}{\partial x} + k\kappa_1 I\right).$$

So for the second invariant. To get the third of the above invariants, multiply the first by a_{p_1+1, q_1}, the second by a_{p_1, q_1+1}, and add. This will give us, since each of these multipliers is itself an invariant — for $p_1 + q_1 = n - 1$, (30) — an invariant of degree $k + 1$; and by (30) that invariant will be the third of the expressions above.

IV. CHANGE OF INDEPENDENT VARIABLES; INVARIANTS AND COVARIANTS.

§ 12. *General Properties.*

We come now to change of independent variables and the invariants and covariants of this transformation. A differential expression in the independent variables $x_1, \ldots x_m$ goes over, under the change of variables

$$\xi_i = \xi_i(x_1, \ldots x_m), \qquad i = 1, 2, \ldots m,$$

into another of the same order. With regard to the coefficients of the latter, which we may call $\bar{a}$, let us note, in the general case, certain facts, sufficient for our purposes.

Any derivative of order k of u with respect to the x's is a polynomial in the derivatives, of order k and less, of u with respect to the ξ's, and in the derivatives of the ξ's with respect to the x's, and is linear in the former set of arguments. These facts follow at once, directly for the first derivatives, by mathematical induction for the higher derivatives, from the formula

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial \xi_j}{\partial x_i}\frac{\partial}{\partial \xi_j}.$$

Hence the $\bar{a}$'s are polynomials in the a's and in the derivatives of the ξ's with respect to the x's, linear in the a's. The derivatives of the $\bar{a}$'s, on the other hand, with respect to the ξ's, are linear polynomials in the a's and their derivatives with respect to the x's, with coefficients polynomials in the derivatives of the ξ's with respect to the x's, the whole divided by a power of the functional determinant of the ξ's with respect to the x's,

$$J \equiv \begin{vmatrix} \dfrac{\partial \xi_1}{\partial x_1} & \cdots\cdots & \dfrac{\partial \xi_1}{\partial x_m} \\ \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ \dfrac{\partial \xi_m}{\partial x_1} & \cdots\cdots & \dfrac{\partial \xi_m}{\partial x_m} \end{vmatrix}.$$

This follows from the formula

$$\frac{\partial}{\partial \xi_i} = \sum_j \frac{\partial x_j}{\partial \xi_i}\frac{\partial}{\partial x_j} = \sum_j \frac{J_{ij}}{J}\frac{\partial}{\partial x_j},$$

J_{ij} being the cofactor in J of $\dfrac{\partial\xi_i}{\partial x_j}$. For the *second* order, the formulas of transformation run as follows:

$$\left.\begin{aligned}\bar{a}_{ij} &= \sum_{k,l} a_{kl}\frac{\partial\xi_i}{\partial x_k}\frac{\partial\xi_j}{\partial x_l},\\ \bar{a}_i &= \sum_{k,l} a_{kl}\frac{\partial^2\xi_i}{\partial x_k\partial x_l} + \sum_k a_k\frac{\partial\xi_i}{\partial x_k},\\ \bar{a} &= a.\end{aligned}\right\} \qquad (36)$$

We next define what we mean by invariants for this transformation.

Definition. By an invariant for a change of independent variables is meant a function of the a's and their derivatives with respect to the x's such that the same function of the $\bar{a}$'s and their corresponding derivatives with respect to the ξ's is equal, by virtue of the formulas of transformation, to the original function multiplied by a power of J, the functional determinant of the ξ's with respect to the x's:

$$I(\bar{a}) = J^w I(a).$$

What we shall have to say about invariants will, in general, as hitherto, refer to polynomial invariants.

As to covariants, besides such as we have already made acquaintance with in the case of change of dependent variable, involving u and its derivatives, we have here a second kind, involving $dx_1, \ldots dx_m$. These two kinds we may distinguish as *covariant differential expressions* and *covariant differential forms* respectively. If we replace u in a covariant differential expression by an *absolute* invariant, it is clear that we shall get an invariant; thus this sort of covariant may be regarded as an operator for deriving invariants; from this point of view it is what is known as a *differential parameter.*

As to the general properties of invariants, we begin with the proposition:

Proposition 19. If we define the weights of the a's and their derivatives as in the case of change of dependent variable, page 19, every invariant is isobaric, of weight w, with respect to any one of the independent variables. Its partial weight, then, with respect to any one of the variables is the same as with respect to any other.

Take the case of two independent variables, x and y. Make the change of variables: $\xi = cx$, $\eta = y$, c being any constant. Then

$$\bar{a}_{pq} = c^p a_{pq}, \qquad \frac{\partial^{i+j}\bar{a}_{pq}}{\partial\xi^i\partial\eta^j} = c^{p-i}\,\frac{\partial^{i+j}a_{pq}}{\partial x^i\partial y^j}, \qquad J = c,$$

so that we have

$$I\left(\dots c^{p-i}\frac{\partial^{i+j}a_{pq}}{\partial x^i \partial y^j},\dots\right) = c^w I\left(\dots \frac{\partial^{i+j}a_{pq}}{\partial x^i \partial y^j},\dots\right),$$

an equation which not only shows that I, if it be a polynomial, is isobaric, but in other cases is commonly used to define what is meant by isobaric with the given system of weights. We shall speak of w as the weight of the invariant even when it is not a polynomial.

The proposition holds also for covariants if, in the case of covariant differential expressions, we attribute to $\frac{\partial^{\alpha+\dots}u}{\partial x_i^{\alpha+\dots}}$ the weight, with respect to x_i, $-\alpha$, and if, in the case of covariant differential forms, we attribute to dx_i the weight one, to dx_j, $j \neq i$, the weight zero, with respect to x_i.

Proposition 20. An invariant may or may not be homogeneous; but if not, it is a mere sum of invariants which are homogeneous.

This is the counterpart of proposition 5, page 19, and the proof is similar in the two cases; for, as noted above, page 40, the $\bar{a}$'s and their derivatives are linear in the a's and their derivatives. So that if we represent by $G_n(a)$ the terms of $I(a)$ of degree n, the corresponding part of $I(\bar{a})$, namely $G_n(\bar{a})$, will be of degree n in the a's and their derivatives.

This proposition may be extended to both kinds of covariants, for the $d\xi$'s are linear in the dx's; and again, as also noted above, the derivatives of u with respect to the x's are linear in the derivatives of u with respect to the ξ's; and this statement may evidently be reversed.

§ 13. *Particular Invariants and Covariants.*

For a differential expression of the *second* order,

$$L(u) = \sum_{i,j} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i a_i \frac{\partial u}{\partial x_i} + au,$$

certain simple invariants and covariants may be deduced by the following considerations.

$$\frac{\partial}{\partial x_i} = \sum_k \frac{\partial \xi_k}{\partial x_i}\frac{\partial}{\partial \xi_k}, \qquad \frac{\partial}{\partial x_j} = \sum_k \frac{\partial \xi_k}{\partial x_j}\frac{\partial}{\partial \xi_k},$$

so that

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_k \frac{\partial^2 \xi_k}{\partial x_i \partial x_j}\frac{\partial u}{\partial \xi_k} + \sum_{k,l} \frac{\partial \xi_k}{\partial x_i}\frac{\partial \xi_l}{\partial x_j}\frac{\partial^2 u}{\partial \xi_k \partial \xi_l}.$$

On the other hand,

$$\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} = \sum_{k,l} \frac{\partial \xi_k}{\partial x_i}\frac{\partial \xi_l}{\partial x_j}\frac{\partial u}{\partial \xi_k}\frac{\partial u}{\partial \xi_l}.$$

It appears thus that the coefficient of $\frac{\partial^2 u}{\partial \xi_k \partial \xi_l}$ in $\frac{\partial^2 u}{\partial x_i \partial x_j}$ is the same as the coefficient of $\frac{\partial u}{\partial \xi_k}\frac{\partial u}{\partial \xi_l}$ in $\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}$. Now in calculating the $\bar{a}$'s with *two* subscripts, $\bar{a}_{ij}$, we are not concerned with the first derivatives of u with respect to the x's or the ξ's; so that the $\bar{a}_{ij}$'s are expressed in terms of the a_{ij}'s, in the case of $L(u)$ under a change of independent variables, by the same formulas as for the expression $\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}$ under the same change of variables, that is, as for the quadratic algebraic form $\sum_{i,j} a_{ij} z_i z_j$ under the linear transformation

$$z_i = \sum_k \frac{\partial \xi_k}{\partial x_i} z_k',$$

a linear transformation whose determinant, as we note, is J.

Now the discriminant

$$A \equiv \begin{vmatrix} a_{11} & \dots & a_{1m} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{m1} & \dots & a_{mm} \end{vmatrix}$$

is a relative invariant of weight two of the algebraic form. A is therefore also a relative invariant of weight two of the differential expression, $L(u)$. We note that A is also an invariant, for change of dependent variable, of the differential equation.

Again, if $v_1, \dots v_m, w_1, \dots w_m$ be two sets of variables contragredient to the $\frac{\partial u}{\partial x}$'s, then

$$- \begin{vmatrix} a_{11} & \dots & a_{1m} & v_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \dots & a_{mm} & v_m \\ w_1 & \dots & w_m & 0 \end{vmatrix} = \sum_{i,j} A_{ij} v_i w_j, \qquad (37)$$

A_{ij} being the cofactor in A of a_{ij}, is invariant of weight two of the algebraic form, and therefore of $L(u)$ also. Now the differentials of the x's are such contragredient variables; so that, if $dx_1, \ldots dx_m$, $\delta x_1, \ldots \delta x_m$ be two sets of differentials, the expressions

$$\sum_{i,j} A_{ij}\, dx_i\, dx_j, \tag{38}$$

$$\sum_{i,j} A_{ij}\, dx_i\, \delta x_j \tag{39}$$

are covariants of weight two. Their coefficients, the A_{ij}'s, are invariants, for change of dependent variable, of the differential equation. Similarly the $(m + p)$-rowed determinant formed by bordering A with p rows and p columns, each of which consists of a set of differentials, is a covariant of weight two.

In the course of the work above we have proved, though we did not at the moment note the fact, that

$$\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \tag{40}$$

is an absolute covariant. The analogous covariant exists for differential expressions of the nth order. For take the terms of $L(u)$ involving derivatives of the nth order, and form an expression $C(u)$ by substituting, for $\dfrac{\partial^n u}{\partial x_1{}^{\beta_1} \ldots \partial x_m{}^{\beta_m}}$, $\left(\dfrac{\partial u}{\partial x_1}\right)^{\beta_1} \ldots \left(\dfrac{\partial u}{\partial x_m}\right)^{\beta_m}$; then $C(u)$ is the covariant in question. For it is easily established by mathematical induction, that the coefficient of $\dfrac{\partial^{\gamma_1+\cdots+\gamma_m} u}{\partial \xi_1{}^{\gamma_1} \ldots \partial \xi_m{}^{\gamma_m}}$ in $\dfrac{\partial^{\beta_1+\cdots+\beta_m} u}{\partial x_1{}^{\beta_1} \ldots \partial x_m{}^{\beta_m}}$ is the same as the coefficient of $\left(\dfrac{\partial u}{\partial \xi_1}\right)^{\gamma_1} \cdots \left(\dfrac{\partial u}{\partial \xi_m}\right)^{\gamma_m}$ in $\left(\dfrac{\partial u}{\partial x_1}\right)^{\beta_1} \cdots \left(\dfrac{\partial u}{\partial x_m}\right)^{\beta_m}$. Whence it follows, just as for the second order, that $C(u)$ is an absolute covariant. $C[f(x_1, \ldots x_m)]$ is also invariant for change of dependent variable, as well as for multiplication of $L(u)$ by ϕ. For $m = 3$, if f satisfy $C(f) = 0$, $f =$ constant is the equation of the characteristic surfaces of $L(u) = 0$. See Sommerfeld in the Encyklopädie der Mathematischen Wissenschaften, II A7c, Nr. 15. The substitution in $C(u)$ for u of an absolute invariant yields an absolute invariant. Since the coefficient of u in $L(u)$, say a, is an absolute invariant, so then also is $C(a)$. For an ordinary differential expression $C(a)$ reduces to $a_n \left(\dfrac{da}{dx}\right)^n$; so that $C(a)$ is, in a certain sense, the analogue for a partial differential expression of the obvious invariant $\frac{da}{dx}$ of an ordinary differential expression. For $n = 2$,

$$C(a) = \sum_{i,j} a_{ij} \frac{\partial a}{\partial x_i} \frac{\partial a}{\partial x_j}.$$

§ 14. *Reduction to Canonical Form of an Ordinary Differential Expression.*

We may obtain, in the case of an ordinary differential expression, a system of rational invariants in terms of which all others may be rationally expressed, by the same device as that employed, § 9, for change of dependent variable. For let the change of variable, $\xi = \chi(x)$, reduce the ordinary differential expression

$$L(u) = a_n u^{(n)} + \dots + a_0 u$$

to a canonical form with coefficients $\bar{A}$. We are to have

$$0 = \bar{A}_{n-1} = (\chi')^{n-2} \left[\frac{n\,(n-1)}{2} \chi''\, a_n + \chi'\, a_{n-1} \right],$$

or

$$\chi'' = -\frac{2}{n\,(n-1)} \frac{a_{n-1}}{a_n} \chi'. \tag{41}$$

Hence any derivative of χ is χ' times a rational function of the a's and their derivatives, and it follows that $\bar{A}_{n-k}$ is $(\chi')^{n-k}$ times such a function. For let $L(u)$ go over under any transformation, $\xi = \phi(x)$, into an expression with coefficients $\bar{a}$. Then the formula

$$\frac{d^k u}{dx^k} = \sum_{l=1}^{k} f_l \frac{d^l u}{d\xi^l},$$

f_l being a polynomial, homogeneous of degree l, in the derivatives of ϕ, may be established by mathematical induction. Hence $\bar{a}_l$ is not only linear in the a's, but homogeneous of degree l in the derivatives of ξ. We have, then, that $\bar{A}_{n-k} = (\chi')^{n-k} J_{n-k}(a)$, J_{n-k} being a rational function. It follows, just as in the similar case of § 9, that J_{n-k} is an invariant of weight $n - k$.

Now let I be any invariant of weight w. Then

$$(\chi')^w I\,(a_n, a_n', \dots a_{n-1}, a_{n-1}', \dots a_{n-k}, \dots \overset{(l)}{a}_{n-k}, \dots)$$

$$= I\left(\bar{A}_n, \frac{d\bar{A}_n}{d\xi}, \dots 0, 0, \dots \bar{A}_{n-k}, \dots \frac{d^l \bar{A}_{n-k}}{d\xi^l}, \dots\right),$$

which, since I is isobaric, is equal to

$$(\chi')^w I\left(\frac{\bar{A}_n}{(\chi')^n}, \frac{1}{(\chi')^{n-1}}\frac{d\bar{A}_n}{d\xi}, \ldots 0, 0, \ldots \frac{\bar{A}_{n-k}}{(\chi')^{n-k}}, \ldots \frac{1}{(\chi')^{n-k-l}}\frac{d^l\bar{A}_{n-k}}{d\xi^l}, \ldots\right)$$
$$= (\chi')^w I\ (J_n, J_{n1}, \ldots 0, 0, \ldots J_{n-k}, \ldots J_{n-k,\,l}, \ldots),$$

if we put

$$\frac{1}{(\chi')^{n-k-l}}\frac{d^l\bar{A}_{n-k}}{d\xi^l} = J_{n-k,\,l}.$$

When we have proved that $J_{n-k,\,l}$ is, like J_{n-k}, a rational invariant, and that it is of weight $n-k-l$, we shall, then, have the proposition:

Proposition 21. Every invariant is a function of the rational invariants

$$J_{n-k} = \frac{\bar{A}_{n-k}}{(\chi')^{n-k}}, \qquad J_{n-k,\,l} = \frac{1}{(\chi')^{n-k-l}}\frac{d^l\bar{A}_{n-k}}{d\xi^l},$$

of weights $n-k$, $n-k-l$ respectively. Here the $\bar{A}$'s are the coefficients of the canonical form into which $L(u)$ goes over, if χ satisfy (41), under $\xi = \chi(x)$.

$$I\ (a_n, a_n', \ldots a_{n-1}, a_{n-1}', \ldots a_{n-k}, \ldots \overset{(l)}{a}_{n-k}, \ldots)$$
$$= I(J_n, J_{n1}, \ldots 0,\ 0, \ldots J_{n-k}, \ldots J_{n-k,\,l}, \ldots).$$

In particular, if I be a polynomial, it is a polynominal in these invariants as well.

The simplest of the invariants in question are:

$$J_n = a_n.$$

$$J_{n1} = a_n' - \frac{2}{n-1}\,a_{n-1}.$$

$$J_{n2} = \frac{n(n-1)\,a_n\,a_n'' - 2n\,a_n\,a_{n-1}' - 2(n-1)\,a_n'\,a_{n-1} + 4a_{n-1}^2}{n(n-1)\,a_n}.$$

$$J_{n-2} =$$

$$\frac{6n(n-1)a_na_{n-2}+2n(n-1)(n-2)(a_n'a_{n-1}-a_na_{n-1}')-(n-2)(3n-1)a_{n-1}^2}{6n(n-1)a_n}$$

We shall find later invariants of a partial differential expression of the second order analogous to J_{n1} and J_{n2}.

It remains to prove that $J_{n-k,l}$ is a rational invariant of weight $n-k-l$. Since

$$J_{n-k,l+1} = \frac{1}{(\chi')^{n-k-l-1}} \frac{d[(\chi')^{n-k-l}J_{n-k,l}]}{d\xi}$$
$$= J_{n-k,l}' - \frac{2(n-k-l)}{n(n-1)} \frac{a_{n-1}}{a_n} J_{n-k,l},$$

the case in hand comes under the proposition:

Proposition 22. If I be an invariant of weight w, then

$$\frac{1}{a_n}\left[a_n I' - \frac{2w}{n(n-1)} a_{n-1} I\right]$$

is an invariant of weight $w-1$.

This proposition may be proved as follows. The expression in question is equal to

$$\frac{1}{a_n}\left[\frac{na_nI' - wa_n'I}{n} + \frac{w}{n}\left(a_n' - \frac{2}{n-1}a_{n-1}\right)I\right].$$

Here $a_n' - \frac{2}{n-1}a_{n-1} = J_{n1}$, and is shown, by direct calculation, with the help of the formulas

$$\bar{a}_n = (\phi')^n a_n,$$
$$\bar{a}_{n-1} = (\phi')^{n-2}\left(\frac{n(n-1)}{2}\phi'' a_n + \phi' a_{n-1}\right),$$

to be an invariant of weight $n-1$. On the other hand, since I^n/a_n^w is an absolute invariant, its derivative is an invariant of weight -1, that is, $na_nI' - wa_n'I$ is an invariant of weight $w+n-1$.

§ 15. *The Adjoint of the Transformed Differential Expression.*

Proposition 6, page 19, gives us, for a change of dependent variable or a multiplication of $L(u)$ by ϕ, a simple relation between the adjoints of the transformed and the original differential expressions. For a change of independent variables we have the following relation:

Proposition 23. If $L(u)$ and its adjoint $M(v)$ go over, under a change of independent variables, into $\bar{L}(u)$ and $\bar{M}(v)$ respectively, then $\frac{\bar{L}(u)}{J}$ and $\frac{\bar{M}(v)}{J}$ are adjoint. To obtain the adjoint of the transformed differential expression we have, then, to subject $M(v)$ to the following transformations:

$\xi_i = \xi_i(x_1, \ldots x_m), \qquad i = 1, 2, \ldots m;$

multiplication by $\frac{1}{J}$;

$v = J \cdot v_1.$

Proof.[7] Make the change of variables in question in Lagrange's Identity,

$$vL(u) - uM(v) = \sum_i \frac{\partial S_i}{\partial x_i},$$

where, as we remember, the S's are bilinear in u, v, and their derivatives of orders up to the $(n-1)$st. Then the S's go over into expressions $\bar{S}$ bilinear in u, v, and their derivatives, with regard to the ξ's, of orders up to the $(n-1)$st, and we have

$$v\bar{L}(u) - u\bar{M}(v) = \sum_i \frac{\partial \bar{S}_i}{\partial x_i} = \sum_{i,j} \frac{\partial \xi_j}{\partial x_i}\frac{\partial \bar{S}_i}{\partial \xi_j}.$$

If we divide this equation through by J, we shall find that we may, without altering the value of the right side, put everything on that side under the signs of differentiation with regard to the ξ's, thus getting

$$v\frac{\bar{L}(u)}{J} - u\frac{\bar{M}(v)}{J} = \sum_j \frac{\partial}{\partial \xi_j}\left(\frac{1}{J}\sum_i \frac{\partial \xi_j}{\partial x_i}\bar{S}_i\right).$$

Here we have, between $\frac{\bar{L}(u)}{J}$ and $\frac{\bar{M}(v)}{J}$, an identity of the form of

---

[7] The proposition in the text is given by du Bois-Reymond in the article Crelle, vol. **104**, already referred to in the note on page 12. His proof, which is based on the Green's Theorem, or integral form of Lagrange's Identity, runs essentially as follows:

$$\int\cdots\int[vL(u) - uM(v)]dx_1 \ldots dx_m = U,$$

U consisting of terms with less than m integrations. But

$$\int\cdots\int[vL(u) - uM(v)]dx_1 \ldots dx_m = \int\cdots\int[v\bar{L}(u) - u\bar{M}(v)]\frac{1}{J}\,d\xi_1\ldots d\xi_m,$$

and so also we may transform U to, say, $\bar{U}$. This gives us

$$\int\cdots\int\left[v\frac{\bar{L}(u)}{J} - u\frac{\bar{M}(v)}{J}\right]d\xi_1\ldots d\xi_m = \bar{U},$$

from which relation, of the form of a Green's Theorem, we infer, just as from a relation of the form of Lagrange's Identity, that $\frac{\bar{L}(u)}{J}$ and $\frac{\bar{M}(v)}{J}$ are adjoint. I have preferred to base my proof on Lagrange's Identity itself.

Lagrange's Identity, and these two expressions are, therefore, in accordance with the proposition noted on page 16, mutually adjoint.

It remains, then, to prove that

$$\sum_{i,j} \frac{1}{J} \frac{\partial \xi_j}{\partial x_i} \frac{\partial \bar{S}_i}{\partial \xi_j} = \sum_{i,j} \frac{\partial}{\partial \xi_j} \left(\frac{1}{J} \frac{\partial \xi_j}{\partial x_i} \bar{S}_i \right),$$

or, what is the same thing, that

$$\sum_{i,j} \bar{S}_i \frac{\partial}{\partial \xi_j} \left(\frac{1}{J} \frac{\partial \xi_j}{\partial x_i} \right) = 0.$$

Now the coefficient of $\bar{S}_i$ in this equation vanishes. For $\frac{1}{J} \frac{\partial \xi_j}{\partial x_i}$ is equal to ι_{ij}, the cofactor in

$$\frac{1}{J} \equiv \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \cdots & \frac{\partial x_1}{\partial \xi_m} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \frac{\partial x_m}{\partial \xi_1} & \cdots & \frac{\partial x_m}{\partial \xi_m} \end{vmatrix}$$

of $\frac{\partial x_i}{\partial \xi_j}$. So that the coefficient in question, viz., $\sum_j \frac{\partial}{\partial \xi_j} \left(\frac{1}{J} \frac{\partial \xi_j}{\partial x_i} \right)$, is equal to $\sum_j \frac{\partial \iota_{ij}}{\partial \xi_j}$, and this expression vanishes. For $\frac{\partial \iota_{ij}}{\partial \xi_j}$ is the sum of the m—1 determinants obtained by substituting in ι_{ij} for the elements of each of its columns in turn the derivatives, with regard to ξ_j, of the elements of that column. Consider any one of these m—1 determinants,

$$(-1)^{i+j} \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \cdots & \frac{\partial x_1}{\partial \xi_{k-1}} & \frac{\partial^2 x_1}{\partial \xi_j \partial \xi_k} & \frac{\partial x_1}{\partial \xi_{k+1}} & \cdots & \frac{\partial x_1}{\partial \xi_{j-1}} & \frac{\partial x_1}{\partial \xi_{j+1}} & \cdots & \frac{\partial x_1}{\partial \xi_m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial x_{i-1}}{\partial \xi_1} & \cdots & \frac{\partial x_{i-1}}{\partial \xi_{k-1}} & \frac{\partial^2 x_{i-1}}{\partial \xi_j \partial \xi_k} & \frac{\partial x_{i-1}}{\partial \xi_{k+1}} & \cdots & \frac{\partial x_{i-1}}{\partial \xi_{j-1}} & \frac{\partial x_{i-1}}{\partial \xi_{j+1}} & \cdots & \frac{\partial x_{i-1}}{\partial \xi_m} \\ \frac{\partial x_{i+1}}{\partial \xi_1} & \cdots & \frac{\partial x_{i+1}}{\partial \xi_{k-1}} & \frac{\partial^2 x_{i+1}}{\partial \xi_j \partial \xi_k} & \frac{\partial x_{i+1}}{\partial \xi_{k+1}} & \cdots & \frac{\partial x_{i+1}}{\partial \xi_{j-1}} & \frac{\partial x_{i+1}}{\partial \xi_{j+1}} & \cdots & \frac{x_{i+1}}{\partial \xi_m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial x_m}{\partial \xi_1} & \cdots & \frac{\partial x_m}{\partial \xi_{k-1}} & \frac{\partial^2 x_m}{\partial \xi_j \partial \xi_k} & \frac{\partial x_m}{\partial \xi_{k+1}} & \cdots & \frac{\partial x_m}{\partial \xi_{j-1}} & \frac{\partial x_m}{\partial \xi_{j+1}} & \cdots & \frac{\partial x_m}{\partial \xi_m} \end{vmatrix}$$

The same determinant occurs a second time, and a second time only, in $\sum_j \frac{\partial \iota_{ij}}{\partial \xi_j}$. It comes, namely, from ι_{ik} also, if we replace therein the elements of the jth column by their derivatives with regard to ξ_k, — the same determinant, that is, except perhaps as to sign; and it is easily seen that the signs in the two cases are opposite, so that the two determinants cancel each other. Thus the $m\ (m - 1)$ determinants, as the sum of which $\sum_j \frac{\partial \iota_{ij}}{\partial \xi_j}$ may be written, cancel each other in pairs; and the latter expression is, as asserted, zero.

V. CONDITIONS FOR $\phi \cdot L(u)$ BEING $(-1)^n$ TIMES ITS ADJOINT.

§ 16. *The Conditions.*

The remainder of this paper will be devoted to a study of the problem: What are the conditions that a differential expression should possess the property of its being possible, by multiplying it by a suitable function, ϕ, of the independent variable or variables, to make $\phi \cdot L(u)$ equal to $(-1)^n$ times its adjoint?[8] After a discussion of ordinary differential expressions I shall give a complete solution of the problem for partial differential expressions of the second order, obtaining also certain results for those of higher order.

Before attacking the problem, let us notice that the property in question is an invariant property. It is, of course, invariant for a multiplication of $L(u)$ by a function of the independent variables. It is invariant for a change of independent variables. For let $L(u)$ go over, under such a change of variables, into $\bar{L}(u)$. Now $\phi \cdot L(u)$ and $(-1)^n \phi \cdot L(v)$ are adjoint. Therefore, by the proposition last proved, $\frac{1}{J}$ times the transformed of $\phi \cdot L(u)$ and $\frac{1}{J}$ times the transformed of $(-1)^n \phi \cdot L(v)$ are adjoint. That is, $\frac{1}{J} \phi \cdot \bar{L}(u)$ and $(-1)^n \frac{1}{J} \phi \cdot \bar{L}(v)$ are adjoint. That is, $\bar{L}(u)$ can be made equal to $(-1)^n$ times its adjoint by multiplying it by $\frac{1}{J}\phi$. In the same way, by making use of proposition 6, page 19, we may show that the property in question is invariant for a change of dependent variable; that is, that if $L(u)$ go over, under $u = \psi \cdot \eta$, into $\Lambda(\eta)$, then $\Lambda(\eta)$ may be made equal to $(-1)^n$ times its adjoint by multiplying it by $\phi\psi$. That property, then, persists under all these transformations. In parallelism with this fact, the conditions we shall obtain for its existence are the vanishing of expressions invariant under all these transformations.

---

[8] This problem is solved, in the case of partial differential expressions of the second order, in *two* independent variables, by du Bois-Reymond in the article referred to in the last note. The fact that the expression, whose vanishing forms the condition for the possibility of a solution, is an invariant, is not, however, noticed.

Taking first the case of *ordinary* differential expressions, let us begin with those of the second order. The condition that

$$L(u) = a_{11}u'' + a_1u' + au$$

should be self-adjoint is, by (10), page 10, $a_1 = a_{11}'$. The condition that $\phi \cdot L(u)$ should be self-adjoint is, therefore,

$$\phi a_1 = \frac{d}{dx}(\phi a_{11}),$$

or

$$a_{11}\phi + (a_{11}' - a_1)\,\phi = 0.$$

It is always possible, then, to make an ordinary differential expression of the second order self-adjoint by multiplying it by a function of x; the latter function has merely to be a solution of the differential equation last written.

We note that, since $\frac{d}{dx}(\phi a_{11}) = \phi a_1$, $\phi \cdot L(u)$ may be written in the form

$$\phi \cdot L(u) = \frac{d}{dx}(Ku') + Gu,$$

where $K = \phi a_{11}$, $G = \phi a$, and ϕ is determined as above. A differential *equation*, then,

$$u'' + pu' + qu = 0,$$

may be thrown into the form

$$\frac{d}{dx}(Ku') + G(u) = 0,$$

where $K = \phi$, $G = \phi q = Kq$, and ϕ is a solution of $\phi' = p\phi$, or, say, $\phi = e^{\int p dx}$. This is Sturm's Normal Form for such an equation.

For ordinary differential equations of the nth order, the solution of our problem will be found in Wilczynski, page 46. The conditions there obtained consist in the vanishing of the so-called linear invariants of odd weight, that is, in Wilczynski's notation, of Θ_3, Θ_5, etc.

To translate into terms of differential expressions we must substitute $\frac{a_{n-k}}{a_n}$ in the Θ's for the coefficient p_{n-k} of the differential equation

$$u^{(n)} + p_{n-1}u^{(n-1)} + \ldots = 0.$$

The expressions so obtained are evidently, like the Θ's, invariants, both for change of dependent and independent variable, of the differential equation $L(u) = 0$.

Next let $L(u)$ be a *partial* differential expression and of the second order. $\phi \cdot L(u)$ is to be self-adjoint. Necessary and sufficient conditions thereto are, by (10), page 10,

$$\phi a_i = \sum_j \frac{\partial(\phi a_{ij})}{\partial x_j},$$

or

$$\sum_j a_{ij} \frac{\partial \log \phi}{\partial x_j} = a_i - \sum_j \frac{\partial a_{ij}}{\partial x_j}, \qquad i = 1, 2, \ldots m.$$

If

$$A \equiv \begin{vmatrix} a_{11} & \cdots & a_{1m} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{m1} & \cdots & a_{mm} \end{vmatrix}$$

is not zero, we may solve these equations, and get

$$\frac{\partial \log\phi}{\partial x_i} = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & a_1 - \sum_j \frac{\partial a_{1j}}{\partial x_j} & a_{1,i+1} & \cdots & a_{1m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdots & a_{m,i-1} & a_m - \sum_j \frac{\partial a_{mj}}{\partial x_j} & a_{m,i+1} & \cdots & a_{mm} \end{vmatrix}}{A} = L_i, \tag{42}$$

let us say. Necessary and sufficient conditions that these equations should have a solution, $\log \phi$, are

$$\frac{\partial L_i}{\partial x_j} - \frac{\partial L_j}{\partial x_i} = 0, \qquad i = 1, 2, \ldots m.$$

The expressions on the left are absolute invariants, for change of dependent variable, of the equation $L(u) = 0$. For if we refer to (24) and (27), pages 31–32, we shall find that $L_i = \lambda_i - \kappa_i$; so that

$$\frac{\partial L_i}{\partial x_j}-\frac{\partial L_j}{\partial x_i}=\frac{\partial(\lambda_i-\kappa_i)}{\partial x_j}-\frac{\partial(\lambda_j-\kappa_j)}{\partial x_i}=\left(\frac{\partial\lambda_i}{\partial x_j}-\frac{\partial\lambda_j}{\partial x_i}\right)-\left(\frac{\partial\kappa_i}{\partial x_j}-\frac{\partial\kappa_j}{\partial x_i}\right),$$

that is, is equal, page 32, to the difference of an absolute invariant and the adjoint invariant.

The expressions $\frac{\partial L_i}{\partial x_j}-\frac{\partial L_j}{\partial x_i}$ are *not*, on the other hand, except in the case of *two* independent variables, invariants for a change of independent variables. They are, however, the coefficients of what, to extend somewhat the definition of that term, we may call a covariant, viz., $\sum_{i,j}\left(\frac{\partial L_i}{\partial x_j}-\frac{\partial L_j}{\partial x_i}\right)dx_i\delta x_j$, where the dx's and δx's are two independent systems of differentials. Reserving for the moment, until we have discussed partial differential expressions of the nth order, the proof that the expression above is a covariant, we may state the solution of our problem, for the case in hand, as follows:

Proposition 24. A necessary and sufficient condition for the possibility of making a differential expression of the second order self-adjoint by multiplying it by a function of the independent variables is, if the invariant A does not vanish, the identical vanishing of the expression

$$\sum_{i,j}\left(\frac{\partial L_i}{\partial x_j}-\frac{\partial L_j}{\partial x_i}\right)dx_i\delta x_j, \tag{43}$$

the L's being defined by (42). The coefficients of this expression, $\frac{\partial L_i}{\partial x_j}-\frac{\partial L_j}{\partial x_i}$, are absolute invariants, for change of dependent variable, of the differential equation, and the expression itself is absolutely invariant for change of independent variables.

Let us look now for a moment at the case of partial differential expressions of the nth order. We take, as usual, for illustration, two independent variables. In order, first, that the coefficients of *the* $(n-1)$*st derivatives* in $\phi\cdot L(u)$ should be $(-1)^n$ times the corresponding coefficients of its adjoint, we must, by (15), page 14, have

$$na_{p+1,q}\frac{\partial\log\phi}{\partial x}+na_{p,q+1}\frac{\partial\log\phi}{\partial y}=2\,a_{pq}-n\left(\frac{\partial a_{p+1,q}}{\partial x}+\frac{\partial a_{p,q+1}}{\partial y}\right),$$
$$p+q=n-1,\qquad p=0,1,\ldots(n-1). \tag{44}$$

If these equations are to be solvable algebraically for $\frac{\partial\log\phi}{\partial x}$, $\frac{\partial\log\phi}{\partial y}$, it is necessary that all three-rowed determinants of the matrix

$$\left\| \begin{array}{lll} a_{n0} & a_{n-1,1} & 2\,a_{n-1,0} - n\left(\dfrac{\partial a_{n0}}{\partial x} + \dfrac{\partial a_{n-1,1}}{\partial y}\right) \\ a_{n-1,1} & a_{n-2,2} & 2\,a_{n-2,1} - n\left(\dfrac{\partial a_{n-1,1}}{\partial x} + \dfrac{\partial a_{n-2,2}}{\partial y}\right) \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ a_{1,n-1} & a_{0n} & 2\,a_{0,n-1} - n\left(\dfrac{\partial a_{1,n-1}}{\partial x} + \dfrac{\partial a_{0n}}{\partial y}\right) \end{array} \right\|$$

should vanish. These three-rowed determinants are invariants, for change of dependent variable, of the differential equation. For any one of them may be written as $I + (-1)^{n-1} J$, where I is an invariant of the form (28), page 33, and J is the adjoint invariant (29).

If these conditions are fulfilled, we may solve for $\dfrac{\partial \log \phi}{\partial x}$, $\dfrac{\partial \log \phi}{\partial y}$ any two of the equations (44):

$$\frac{\partial \log \phi}{\partial x} = \frac{\begin{vmatrix} 2\,a_{p_1 q_1} - n\left(\dfrac{\partial a_{p_1+1,q_1}}{\partial x} + \dfrac{\partial a_{p_1,q_1+1}}{\partial y}\right) & a_{p_1,q_1+1} \\ 2\,a_{p_2 q_2} - n\left(\dfrac{\partial a_{p_2+1,q_2}}{\partial x} + \dfrac{\partial a_{p_2,q_2+1}}{\partial y}\right) & a_{p_2,q_2+1} \end{vmatrix}}{n\,A} = L_1,$$

$$\frac{\partial \log \phi}{\partial y} = \frac{\begin{vmatrix} a_{p_1+1,q_1} & 2\,a_{p_1 q_1} - n\left(\dfrac{\partial a_{p_1+1,q_1}}{\partial x} + \dfrac{\partial a_{p_1,q_1+1}}{\partial y}\right) \\ a_{p_2+1,q_2} & 2\,a_{p_2 q_2} - n\left(\dfrac{\partial a_{p_2+1,q_2}}{\partial x} + \dfrac{\partial a_{p_2,q_2+1}}{\partial y}\right) \end{vmatrix}}{n\,A} = L_2,$$

$$A = \begin{vmatrix} a_{p_1+1,q_1} & a_{p_1,q_1+1} \\ a_{p_2+1,q_2} & a_{p_2,q_2+1} \end{vmatrix}, \qquad p_i + q_i = n - 1.$$

And the necessary and sufficient condition for the existence of a solution, $\log \phi$, is $\dfrac{\partial L_1}{\partial y} - \dfrac{\partial L_2}{\partial x} = 0$. Here the expression on the left is an invariant, for change of dependent variable, of the differential equation. For $L_1 = \lambda_1 - \kappa_1$, $L_2 = \lambda_2 - \kappa_2$, the κ's and λ's being given by (30) and (32 a), page 33; and $\dfrac{\partial L_1}{\partial y} - \dfrac{\partial L_2}{\partial x}$ is, therefore, the difference between an invariant and the adjoint invariant.

We shall carry the solution of the problem no further. To complete that solution we should next have to go on and write down the conditions that the coefficients of the derivatives of the $(n-3)$d, $(n-5)$th, etc., orders in $\phi \cdot L(u)$ should be $(-1)^n$ times the corresponding coefficients of its adjoint. (By proposition 4, page 15, we need merely consider the orders $n-k$, where k is odd.) These conditions would, by (15), page 14, be the vanishing of expressions bilinear in the a's and their derivatives and in ϕ and its derivatives; that is, after substitution for the derivatives of ϕ from the equations $\frac{\partial \phi}{\partial x} = L_1\phi$, $\frac{\partial \phi}{\partial y} = L_2\phi$, and from the equations obtained from these by differentiation, and after division by ϕ, of rational functions of the a's and their derivatives. And the question would suggest itself as to whether these latter were invariants.

§ 17. *The Covariant* $\sum_{i,j} \left(\frac{\partial L_i}{\partial x_j} - \frac{\partial L_j}{\partial x_i}\right) dx_i \delta x_j$.

We return now to the proof that the expression (43), page 53, is a covariant for change of independent variables. A proof of this fact is to be found in a paper by E. Cotton, *Sur les Invariants Différentiels de quelques Equations linéaires aux dérivées partielles du second ordre,* in the Annales de l'Ecole Normale, 3e série, vol. 17 (1900), pages 211–244. Cotton's methods are based on the theory of quadratic differential forms. It is perhaps worth while to obtain the result we are interested in independently of that theory, as may be done with no great difficulty. I shall therefore give such a proof, following in general the steps by which Cotton reaches his result. I retain in part his notation. Further, a dash over an expression shall indicate that it is the same function of the $\bar{a}$'s, the coefficients of the transformed differential expression, that the expression without the dash is of the a's.

First, then, the expression

$$\Delta_2 u = \sqrt{A} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\frac{a_{ij}}{\sqrt{A}} \frac{\partial u}{\partial x_j}\right)$$

is an absolute covariant. Here A stands, as usual, for the determinant of the a_{ij}'s, an invariant, as we know, of weight two. The proof goes as follows. Making use of the formulas (36), page 41, for the $\bar{a}$'s, we get

$$
\begin{aligned}
\overline{\Delta_2 u} &= \sqrt{\bar{A}} \sum_{i,j} \frac{\partial}{\partial \xi_i}\left(\frac{\bar{a}_{ij}}{\sqrt{\bar{A}}}\frac{\partial u}{\partial \xi_j}\right) \\
&= J\sqrt{A} \sum_{i,j} \frac{\partial}{\partial \xi_i}\left[\sum_{k,l} \frac{a_{kl}}{J\sqrt{A}} \frac{\partial \xi_i}{\partial x_k} \frac{\partial \xi_j}{\partial x_l} \frac{\partial u}{\partial \xi_j}\right] \\
&= J\sqrt{A} \sum_{i} \frac{\partial}{\partial \xi_i}\left[\sum_{k,l} \frac{a_{kl}}{J\sqrt{A}} \frac{\partial \xi_i}{\partial x_k} \left(\sum_{j} \frac{\partial u}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_l}\right)\right] \\
&= J\sqrt{A} \sum_{i} \frac{\partial}{\partial \xi_i}\left[\sum_{k,l} \frac{a_{kl}}{J\sqrt{A}} \frac{\partial \xi_i}{\partial x_k} \frac{\partial u}{\partial x_l}\right] \\
&= J\sqrt{A} \sum_{i,k,l} \left[\frac{\partial \xi_i}{\partial x_k} \frac{\partial}{\partial \xi_i}\left(\frac{a_{kl}}{J\sqrt{A}} \frac{\partial u}{\partial x_l}\right)\right] \\
&\qquad + \sum_{k,l} \left[a_{kl} \frac{\partial u}{\partial x_l} \sum_{i,j} \left(\frac{\partial^2 \xi_i}{\partial x_j \partial x_k} \frac{\partial x_j}{\partial \xi_i}\right)\right].
\end{aligned}
$$

The first half of this expression

$$
\begin{aligned}
&= J\sqrt{A} \sum_{k,l} \frac{\partial}{\partial x_k}\left(\frac{a_{kl}}{J\sqrt{A}} \frac{\partial u}{\partial x_l}\right) \\
&= \sqrt{A} \sum_{k,l} \frac{\partial}{\partial x_k}\left(\frac{a_{kl}}{\sqrt{A}} \frac{\partial u}{\partial x_l}\right) - \frac{1}{J} \sum_{k,l} a_{kl} \frac{\partial u}{\partial x_l} \frac{\partial J}{\partial x_k} \\
&= \Delta_2 u - \frac{1}{J} \sum_{k,l} a_{kl} \frac{\partial u}{\partial x_l} \frac{\partial J}{\partial x_k}.
\end{aligned}
$$

Now

$$\frac{\partial J}{\partial x_k} = \sum_{i,j} \frac{\partial^2 \xi_i}{\partial x_j \partial x_k} J_{ij},$$

if J_{ij} be the cofactor, in J, of $\frac{\partial \xi_i}{\partial x_j}$. Further

$$J_{ij} = J \frac{\partial x_j}{\partial \xi_i};$$

so that we have finally

$$\overline{\Delta_2 u} = \Delta_2 u - \sum_{i,j,k,l} a_{kl} \frac{\partial u}{\partial x_l} \frac{\partial^2 \xi_i}{\partial x_j \partial x_k} \frac{\partial x_j}{\partial \xi_i} + \text{the same quadruple sum;}$$

that is,

$$\overline{\Delta_2 u} = \Delta_2 u. \qquad Q.\ E.\ D.$$

$\Delta_2 u$, then, or, written in expanded form,

$$\Delta_2 u = \sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i,j} \frac{\partial u}{\partial x_i} \left(\frac{\partial a_{ij}}{\partial x_j} - \frac{1}{2A} a_{ij} \frac{\partial A}{\partial x_j} \right),$$

is an absolute covariant. We notice that the terms involving second derivatives are identical in $\Delta_2 u$ and in $L(u)$, so that the latter may be written

$$L(u) = \Delta_2 u + \sum_i d_i \frac{\partial u}{\partial x_i} \qquad au,$$

where

$$d_i = a_i - \sum_j \frac{\partial a_{ij}}{\partial x_j} + \frac{1}{2A} \sum_j a_{ij} \frac{\partial A}{\partial x_j}. \tag{45}$$

Similarly, the transformed differential expression, $\bar{L}(u)$, may be written

$$\bar{L}(u) = \overline{\Delta_2 u} + \sum_i \bar{d}_i \frac{\partial u}{\partial \xi_i} + \bar{a} u.$$

Now since, when $L(u)$ goes over into $\bar{L}(u)$, $\Delta_2 u$, au go over into $\overline{\Delta_2 u}$, $\bar{a}u$ respectively, it follows that $\sum_i d_i \frac{\partial u}{\partial x_i}$ goes over into $\sum_i \bar{d}_i \frac{\partial u}{\partial \xi_i}$, in other words that it is an absolute covariant.

Hence we conclude that the d's are transformed contragrediently to the $\frac{\partial u}{\partial x}$'s. The expression $\sum_{i,j} A_{ij} d_j dx_i$, then, is of the form of (37), page 43, and is, therefore, a relative covariant of weight two; or

$$\sum_i l_i dx_i \tag{46}$$

is an absolute covariant, if we define l_i by the formula

$$l_i = \frac{1}{A} \sum_j A_{ij} d_j. \tag{47}$$

Since (46) is an absolute covariant, the l's must be transformed contragrediently to the dx's,

$$\bar{l}_i = \sum_k \frac{\partial x_k}{\partial \xi_i} l_k. \tag{48}$$

This being the case, the expression

$$\sum_{i,j} \left(\frac{\partial l_i}{\partial x_j} - \frac{\partial l_j}{\partial x_i} \right) dx_i \delta x_j, \tag{49}$$

where the dx's and δx's are two independent sets of differentials, will be an absolute covariant.

Proof. Consider an expression $\sum_{i,j} c_{ij} dx_i \delta x_j$; and let it go over by our change of variables into $\sum_{i,j} \bar{c}_{ij} d\xi_i \delta\xi_j$. Then for the $\bar{c}$'s we readily calculate the formula,

$$\bar{c}_{pq} = \sum_{i,j} c_{ij} \frac{\partial x_i}{\partial \xi_p} \frac{\partial x_j}{\partial \xi_q}.$$

Now the coefficients, $\frac{\partial l_i}{\partial x_j} - \frac{\partial l_j}{\partial x_i}$, in (49) above are transformed cogrediently with the c's. For we have, from (48),

$$\frac{\partial \bar{l}_p}{\partial \xi_q} = \frac{\partial}{\partial \xi_q}\left(\sum_i \frac{\partial x_i}{\partial \xi_p} l_i\right) = \sum_i \frac{\partial^2 x_i}{\partial \xi_p \, \partial \xi_q} l_i + \sum_{i,j} \frac{\partial x_i}{\partial \xi_p} \frac{\partial x_j}{\partial \xi_q} \frac{\partial l_i}{\partial x_j},$$

$$\frac{\partial \bar{l}_q}{\partial \xi_p} = \sum_i \frac{\partial^2 x_i}{\partial \xi_p \partial \xi_q} l_i + \sum_{i,j} \frac{\partial x_i}{\partial \xi_p} \frac{\partial x_j}{\partial \xi_q} \frac{\partial l_j}{\partial x_i}.$$

Therefore

$$\frac{\partial \bar{l}_p}{\partial \xi_q} - \frac{\partial \bar{l}_q}{\partial \xi_p} = \sum_{i,j} \left(\frac{\partial l_i}{\partial x_j} - \frac{\partial l_j}{\partial x_i}\right) \frac{\partial x_i}{\partial \xi_p} \frac{\partial x_j}{\partial \xi_q},$$

as asserted. Hence it follows that just as we have

$$\sum_{i,j} \bar{c}_{ij} d\xi_i \delta\xi_j = \sum_{i,j} c_{ij} dx_i \delta x_j,$$

so also we have

$$\sum_{i,j} \left(\frac{\partial \bar{l}_i}{\partial \xi_j} - \frac{\partial \bar{l}_j}{\partial \xi_i}\right) d\xi_i \delta\xi_j = \sum_{i,j} \left(\frac{\partial l_i}{\partial x_j} - \frac{\partial l_j}{\partial x_i}\right) dx_i \delta x_j. \qquad Q.E.D.$$

Now the covariant (49) is identical with the expression (43) which we wish to prove a covariant. To establish this identity we need merely to obtain the explicit form of (49). From formulas (47), (45) we get

$$l_i = \frac{1}{A} \sum_j A_{ij} d_j = \frac{1}{A} \sum_j A_{ij} \left(a_j - \sum_k \frac{\partial a_{jk}}{\partial x_k} + \frac{1}{2A} \sum_k a_{jk} \frac{\partial A}{\partial x_k}\right)$$

$$= \frac{1}{A} \sum_j A_{ij} \left(a_j - \sum_k \frac{\partial a_{jk}}{\partial x_k}\right) + \frac{1}{2A^2} \sum_k \left(\frac{\partial A}{\partial x_k} \sum_j a_{kj} A_{ij}\right).$$

The first part of this expression will be seen to be equal to L_i, as defined by (42), page 52. Since, further,

$$\begin{aligned} \sum_j a_{kj} A_{ij} &= 0, \qquad & k \neq i, \\ &= A, \qquad & k = i, \end{aligned}$$

we get finally

$$l_i = L_i + \frac{1}{2A} \frac{\partial A}{\partial x_i}.$$

Hence

$$\frac{\partial l_i}{\partial x_j} - \frac{\partial l_j}{\partial x_i} = \frac{\partial L_i}{\partial x_j} - \frac{\partial L_j}{\partial x_i},$$

and our expression, (43), page 53, is identical with (49), and is therefore an absolute covariant. But this is what we set out to prove.

In the case of two independent variables, $m = 2$, our covariant is

$$\left(\frac{\partial L_1}{\partial y} - \frac{\partial L_2}{\partial x}\right) (dx\delta y - dy\delta x).$$

Here the second factor is itself a covariant of weight one; so that, in this case, the condition of proposition 24, page 53, would be the vanishing of $\frac{\partial L_1}{\partial y} - \frac{\partial L_2}{\partial x}$, which is not only an absolute invariant, for change of dependent variable, of the differential equation, but an invariant, of weight minus one, for change of independent variables as well.

I collect here for reference the covariants that we have come across in the course of our work above, adding a couple of invariants from Cotton's paper.[9]

$$\begin{aligned} \Delta_2 u &= \sqrt{A} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\frac{a_{ij}}{\sqrt{A}} \frac{\partial u}{\partial x_j}\right) \\ &= \sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{i,j} \frac{\partial u}{\partial x_i} \left(\frac{\partial a_{ij}}{\partial x_j} - \frac{1}{2A} a_{ij} \frac{\partial A}{\partial x_j}\right), \end{aligned}$$

$\sum_i d_i \frac{\partial u}{\partial x_i}$, and $\sum_i l_i dx_i$ are absolute covariants; d_i and l_i are defined by (45) and (47).

---

[9] For bibliography, see the note, page 239, of Cotton's article. The invariant, for $m = 2$, $\frac{\partial L_1}{\partial y} - \frac{\partial L_2}{\partial x}$ is also given, in explicit form, by Rivereau in the Bull. de la Soc. Math. de France, **29**, 7 (1901); it is identical, as is easily shown, with what Rivereau calls $2I$.

$$\Delta(l) = \sum_{i,j} a_{ij} l_i l_j,$$

$$\Delta_2(l) = \sum_i \frac{\partial d_i}{\partial x_i} - \frac{1}{2A} \sum_i d_i \frac{\partial A}{\partial x_i},$$

are absolute invariants.

For *one* independent variable, these invariants reduce to $\frac{(2a_1 - a_{11}')^2}{16 a_{11}}$ and $\frac{1}{8} \frac{a_{11}}{2a_1 - a_{11}'} \left(\frac{(2a_1 - a_{11}')^2}{a_{11}} \right)'$ respectively. The first of these is the square of the invariant J_{n1}, page 46, for $n = 2$, divided by $16 a_{11}$; while J_{n2}, for $n = 2$, is $8\,\Delta(l) - 4\,\Delta_2(l)$. Thus we have found, for the second order, invariants of a partial differential expression analogous to the invariants J_{n1}, J_{n2} of an ordinary differential expression.

We shall accept from Cotton the fact that $\Delta_2(l)$ is an absolute invariant. Next as to $\Delta(l)$. Since $\sum_{i,j} \frac{A_{ij}}{A} dx_i dx_j$ is an absolute covariant, — cf. (38), page 44, — any invariant of this quadratic differential form of weight w will be an invariant of $L(u)$ of weight $-w$. Now since, page 57, the l's are contragredient to the dx's,

$$-\begin{vmatrix} \frac{A_{11}}{A} & \cdots & \frac{A_{1m}}{A} & l_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ \frac{A_{m1}}{A} & \cdots & \frac{A_{mm}}{A} & l_m \\ l_1 & \cdots & l_m & 0 \end{vmatrix},$$

that is, $\sum_{i,j} \frac{a_{ij}}{A} l_i l_j$, is an invariant of weight two of the differential form. $\sum_{i,j} a_{ij} l_i l_j$ or $\Delta(l)$ is, then, an absolute invariant of $L(u)$.

CAMBRIDGE, MASS.,
April, 1908.